
C. V. L. CHARLIER

ELEMENTS OF MATHEMATICAL STATISTICS

also

L. v. Bortkiewicz

TABLE OF POISSON'S FREQUENCY FUNCTION

edited and translated by

J. A. Greenwood

ἐγὼ δὲ τοῖς διὰ τῶν εἰκότων τὰς ἀποδείξεις ποιουμένοις λόγοις ξύνοιδα οὖσιν ἀλαζόσι, καὶ ἄν τις αὐτοὺς μὴ φυλάττηται, εὖ μάλα ἐξαπατῶσι, καὶ ἐν γεωμετρίᾳ καὶ ἐν τοῖς ἄλλοις ἅπασιν.
Plato, *Phaedo* 92 D

CAMBRIDGE, MASSACHUSETTS

1947

PHOTO-LITHOGRAPHY BY
SPAULDING-MOSS COMPANY
BOSTON, MASSACHUSETTS, U.S.A.

First Edition 1910
Grunddragen af den matematiska Statistiken
Lund, Statsvetenskaplig Tidskrifts Expedition
Second Edition 1920
Vorlesungen über die Grundzüge der mathematischen Statistik
Lund, Verlag Scientia
Photo-offset in U. S. A.

Preface to the first Edition.

Mathematical statistics is the tool whose help enables the statistician to draw conclusions from his statistical material. As with any other tool, the result and the value of its application depend primarily upon how the director of the work understand the execution of his assignment. This tool can just as easily be misused or perverted as properly used--perhaps more easily--especially since the mathematical apparatus has recently received a sharpness which did not even approximately exist formerly. Mathematical statistics is no automaton, wherein one need only insert the statistical data and then after some mechanical operations read off as from a calculating machine, the result. One is not always certain of obtaining by such mechanical operations the correct answer to the problem.

With this reservation, then, it must be said that mathematical statistics is just as necessary for the statistician as the knife for the surgeon. The statement proper of the question--that is, the formulation of the problem to be solved and likewise the collecting of the statistical data relevant to and shedding light on the question--involves two tasks that demand special technical knowledge in that scientific branch to which the problem belongs. Once, however, the data have been collected and the relevant question formulated, to answer that question is a task that lies wholly in the realm of mathematical statistics.

Mathematical statistics dates, as a science, from the beginning of the eighteenth century. The famed theorem of BERNOULLI remains today the foundation of the structure of statistics. On this is founded the fundamental principle of all statistics: to derive, from statistically large numbers, the laws of complex events, which underlie the fluctuating numerical statistical series. The 'divine order' of SUSSMILCH is perhaps to be taken as the most complete expression of the viewpoint of applied statistics during the time immediately after the publication of the fundamental theorem of BERNOULLI.

The developement which began with BERNOULLI received, principally through the investigations of DE MOIVRE, GAUSS, and LAPLACE, a certain conclusion. The *Théorie analytique des probabilités* of LAPLACE [14] is doubtless the most significant work that has appeared in the realm of mathematical statistics. Unfortunately, ex-

cepting POISSON's elegant developement of some theorems of LAPLACE, the many points of attack found in the work of LAPLACE remained as good as unobserved. Only in very recent years have we opened our eyes to the great collection of undeveloped basic theorems which are entombed in the great work of LAPLACE.

Responsibility for the stagnation which set in at that time in the developement of mathematical statistics rests principally upon GAUSS. This great mathematician believed himself able to prove that fluctuations in the elements of a statistical series--he concerned himself chiefly with series of astronomical and geodetic observations--follow strictly the simple law called after him the Gaussian law of error. Where deviations appeared, he believed that he could attribute them solely to the small number of observations. He stated--on the basis of an erroneous mathematical proof --that the deviations would vanish if only the number of observations were sufficiently large. This theorem pervaded all mathematical statistics of the nineteenth century like an article of faith, and the method of least squares, based on the Gaussian law of error, was, and often still is, considered a definitive solution of the problem of strict scientific treatment of statistical series of observations.

In application of statistics other than to astronomy, QUETELET appears to be him who applied the principles of the Gaussian law of error with greatest success. His doctrine of type is certainly a most important basic theorem of statistics; but one must carefully guard against the presumption that the mathematical type of the statistical object necessarily corresponds to an actual (physical, biological) type. In this last respect QUETELET was guilty of great excesses and thereby, in many respects against his will, brought a discredit upon mathematical statistics from which it has not yet altogether recovered.

Three years after the death of QUETELET appeared the paper of LEXIS, *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft* [15], which contains the first essential progress in mathematical statistics since the days of LAPLACE. LEXIS shews here that statistical events do not altogether follow, as it was customary to assume, the Bernoullian laws of probability. He gives (more completely in other papers) an explanation by mathematical statistics of these deviations and is led thereby to a simple criterion which serves to estimate the intensity of the foreign disturbances to which a statistical event is exposed.

Nevertheless the Gaussian law of error remained unshaken. Then, however, at the end of the nineteenth century, came the breakthrough. Just as has so often been the case, as history shews, with the progress of new scientific truths, here also came the discovery simultaneously or nearly simultaneously from many sides. It is particularly interesting to observe, as signifying

the universality of statistical science, how representatives of the furthest parts of science reached the same goal by more or less different paths. We find among the pioneers the astronomers THIELE and BRUNS, the psychologists FECHNER and LIPPS, the biologists GALTON and PEARSON, the political economists EDGEWORTH and others. Today there is a ferment of new thoughts and discoveries in this field of labour, so that one may well say that mathematical statistics has unfolded in the last decades into a new science.

Among the investigators who together have worked here must be particularly noticed the English mathematician and biologist KARL PEARSON, professor of applied mathematics (mechanics) at the University of London. In a rather large series of papers, principally in the *Philosophical Transactions* under the general title 'Mathematical contributions to the theory of evolution' [18], this outstanding scholar has attacked and solved a class of difficult problems in mathematical statistics; he has moreover succeeded in assembling about him a numerous school of men and women scholars of the most different branches of the science, who work with unflagging zeal at the further development of the ideas of PEARSON.

Although, during my years of statistical study, I was moved to acquaint myself as carefully as possible with the results of PEARSON and his disciples, the exposition of mathematical statistics given here is not directly based on the investigations of PEARSON. Without wishing to undertake a detailed critique of his investigations, which, moreover, I most highly admire, I nevertheless believe it necessary to remark that the methods of PEARSON possess an essential error, which consists in lacking sufficient generality both in the choice of the starting point and in the practical application. His 7 types of error law[1] are unquestionably admirable formulae of interpolation; but they are derived without reference to the genetic developement of such laws of error. Also there is no place in them for the computation of the higher characteristics of statistical series. His theory of correlation has a similar flaw.

My treatment of mathematical statistics is based on two small notices in the *Meddelanden från Lunds Observatorium* [2; 3]. The immediate starting point for deriving the general form of statistical series can be found in LAPLACE's *Théorie analytique,* and I found later [5], that the same basic principles can be extended to the general form of correlation functions. Thus the entire field of mathematical statistics, as far as it is now worked out, receives a unified treatment. Not only is clearness in the mathematical treatment of the problem obtained in this manner, but also it is seen that the resulting formulae for practical numerical computation take an exceedingly simple form. I have convinced myself in

1. <The PEARSON system has been extended to 13 types. --Tr.>

several practical cases that these formulae can be applied and used by persons who have had no opportunity for a special mathematical education.

In the present redaction, which I undertook at the request of Professor FAHLBECK for the *Statsvetenskaplig Tidskrift*, most mathematical derivations are omitted. I will however display the signification of the formulae by means of numerical examples from various fields of applied statistics. My treatment here will principally be only a resume of practical prescriptions for the numerical treatment of statistical series.

Before beginning my account, I feel that a few words should be said about the division of the subject of statistics that I have found necessary for several reasons to introduce--a division which perhaps may appear strange to professional statisticians. I divide statistics into two essentially different parts:

I. Homograde or alternative statistics,

II. Heterograde or qualitative statistics.

These two parts differ in the nature of the original lists, in the manner of portraying the statistical series, and, last and most important, in the problems that the statistician has to solve. In homograde statistics the theorems of BERNOULLI and of LEXIS furnish the ruling concepts. Their use yields conclusions as to the extent of the foreign disturbances which work upon the statistical object, and the task of the statistician is to investigate and distinguish these foreign influences. In heterograde statistics the mathematicsl theory gives no information about the types involved, which must be determined solely from the empirical data.

The connexion between different statistical events is given, in both parts of statistics, by the theory of correlation, which, however, takes on a somewhat different form within each of the two principal parts of statistics.

Preface to the second Edition.

This edition is principally an unchanged translation of the first (Swedish) edition. Chapter XV: 'Abbreviated methods for the computation of the characteristics' and the tables of the functions Q, R, φ_0, φ_3, and φ_4 have been added.

My hearty thanks are due to Messrs MESSOW and BAADE of the Hamburg observatory, who have kindly helped me with the proof-reading, and to the printer, LUTCKE & WULFF, Hamburg, which has most graciously undertaken to print the German edition.

LUND OBSERVATORY, 1 January 1920,

C. V. L. Charlier.

Translator's Preface.

The text of the present work, which was undertaken at the suggestion of Professor W. L. CRUM of Harvard, follows closely the edition of 1920. Footnotes which the translator has added are enclosed in brackets and indicated by the mark --Tr.

The numerical examples have been taken unchanged from the edition of 1920 with one important exception. Both the translator and Professor CRUM deem the introduction of the factor 5 into equation (2) of Chapter X., which CHARLIER explains in the words 'Die Y-Koordinaten sind mit einem beliebig gewahlten Faktor (5) multipliziert, um bei dem Zeichnen der Kurve (auf gewohnlichem quadrierten Papier) *dieselbe* Skala fur beide Koordinaten zu erhalten', unnecessary and likely to confuse the beginning student. This factor has therefore been suppressed and Tables 23, 24, 25, 26, 27, Figures 1, 2, 3, 4, 5, and the diagrams of Art. 62 and of the appendix altered accordingly.

The EDGEWORTH term in φ_6 and its influence on the excess have been added in footnotes.

Table 41 is composed as follows: φ_{-1} and φ_0 are taken from SHEPPARD [19 , II: 2-7]. φ_2, φ_3, and φ_5 were computed by the translator, partly to thirteen places of decimals, partly to ten places, using as basis LOWAN's table of φ_0 [16] and JØRGENSEN's tables of the second, third, and fifth HERMITE polynomials [12 , VI: 196-199 , 202-203], and then rounded to seven places. φ_4

is from CHARLIER's Table V. φ_6 is cut down to four places from JØRGENSEN [12 , V: 178-193].

Table 42 has been checked against KONDO and ELDERTON [20 , II: 2-10]. Eight last-place errors have been corrected.

Table 43 reproduces the tables of BORTKIEWICZ [1, 49-52]. As SOPER [19, p. lxxvj] points out, the fourth digit of BORTKIEWICZ is not always correct; errors found by comparison with SOPER's table [19, LI: 113-117] have been corrected.

Table 44 was newly computed by the translator.

The translator wishes to acknowledge the assistance of Messrs L. R. BROOKS, P. V. BUGSTROM, H. V. DU BOUCHET, MAC KINNON A. CREELEY, CAROLINE NEEF, G. K. TAKAYAMA, and H. A. THOMAS JR.

CAMBRIDGE. MASSACHUSETTS, 19 January 1947.

Part One.

Homograde or alternative Statistics.

Cap. I. Introduction. Definition of homograde and of heterograde Statistics.

The primary data from which statistical numbers are derived comprise a list of individuals for which a certain attribute has been observed and possibly measured. The individuals can be men, animals, plants, or even inanimate or abstract things and phaenomena; the attribute may be a length, a weight, or any other property of the individuals.

The collection of all individuals which are observed in a certain statistical investigation forms a population.

The original collection of observations has been called the original list.

The attribute being studied may in general assume a continuous or discontinuous range of degrees of intensity. If this intensity has been measured, it can be expressed by a multiple or other function of a suitable unit. The intensity measured in this manner is called the degree of the attribute.

The simplest method of constructing an original list is simply to note whether a certain attribute--or possibly a certain degree of this attribute--is present or absent among the individuals which belong to the population. The original list has then the following appearance:

Table 1.

Original list of homograde individuals.

Designation of the individual	The attribute present	The attribute absent
I_1	1	
I_2		1
I_3		1
I_4	1	
I_5	1	

1 It is here assumed that the individuals I_1, I_4, and I_5 possess the attribute in question, but that the individuals I_2 and I_3 do not.

In publishing observations of this kind it is seldom necessary or even useful to give the original list in this original form. The whole population is then divided into groups consisting of, say, s_1, s_2, ..., s_N individuals, so that

$$s_1 + s_2 + s_3 + \dots + s_N = P, \tag{1}$$

where P denotes the total number of individuals in the population and N the number of groups.

If in particular s_1, s_2, s_3, ..., s_N contain the same number ($= s$) of individuals--which is often approximately the case--equation (1) assumes the form

$$N \cdot s = P.$$

Now let us designate the number of individuals in the different groups that possess the attribute in question by m_1, m_2, m_3, ..., m_N. We obtain as the result of observations the series

$$m_1, m_2, m_3, \dots, m_N,$$

which is called a statistical series of homograde quantities.

The numbers m_1, m_2, m_3, ..., m_N are called the elements of the statistical series; s_1, s_2, s_3, ..., s_N are called the numbers of comparison of the elements.

2 When the form of original list is prepared as in Table 1, we have to decide whether the attribute in question is present or absent in each individual. If the attribute in question can occur in only one degree, then all individuals are identical in this respect. This is the case with statistics of the number of men or women in a country, of the number of births, deaths, immigrants, emigrants, school children, suicides, etc. A person is dead or not, newborn or not, suicide or not, and a graduation of these properties is not possible, except under highly artificial assumptions.

It is not however necessary in preparing Table 1 that all the individuals listed in the second column possess the attribute in question in the same degree. Observe *ex.gr.* a population of grown men and arrange them according to their cephalic index (100 times breadth of head divided by length). If all men with an index smaller than, say, 80, are called dolichocephalic, then we can construct an original list like Table 1 by designating the attribute in question as 'dolichocephaly'. Obviously all individuals who are designated as dolichocephalic are not identical with respect to the attribute in question, but in the preparation of the original table the gradation of the attribute can be neglected.

If the individuals all possess an attribute in the same degree or if no attention is paid in the statistical investigation to the different intensity of the attribute in different individuals, it is said that the individuals are homograde. The part of statistics concerned with such individuals is called homograde statistics.

With reference to the manner of preparation of the original list it can also be called alternative statistics.

3 Contrariwise, individuals which possess a certain attribute in different degrees are called heterograde, and that part of statistics concerned with such individuals (quantities) is called heterograde statistics. Instead of this expression the name qualitative statistics may be used.

The customary form of a primary list of heterograde individuals is the form of the table adjacent.

Here the numbers x_1, x_2, x_3, x_4, x_5, etc., shew the intensity (the degree) of the attribute in question in each individual, expressed in a suitable unit (meter, liter, kilogram, year, etc.) The numbers

$$x_1, x_2, x_3, \ldots, x_N$$

now directly form the statistical series, and each x_k is an element of the series.

Table 2.

Original list of heterograde individuals

Designation of the individual	Degree of the attribute
I_1	x_1
I_2	x_2
I_3	x_3
I_4	x_4
I_5	x_5

As examples of statistical attributes which belong to heterograde statistics may be named the duration of life or of an illness, the length, volume, or weight of animals, plants, or inanimate objects, the length of the period of gestation, the colour of the hair or of the eyes, the spectrum of the stars, etc.

Cap. II. The arithmetic Mean.

4 The first task of mathematical statistics is to show how the characteristic properties of a statistical series, or, more generally, of a number of simultaneously considered statistical series can be determined. Experience has taught that in the apparent disorder that a statistical series displays simple laws rule and permit each statistical series or group of simultaneously considered interdependent statistical series to be simply and uniquely characterised.

The numbers that express the essential properties of a statistical series are called in this work the characteristics of the statistical series.

In most cases four of these characteristics are sufficient.

4 For rather small series (= series with a small number [N] of elements) two of them suffice. In exceptional cases one can be satisfied with a single characteristic.

The first four characteristics have received the following names:

1. the medium[1] or arithmetic mean (M);
2. the dispersion[1] (σ);
3. the skewness or asymmetry (S);
4. the excess (E).

I shall assume, here and in several chapters to follow, that those individuals which belong to the original list in Table 1 are divided into N groups, each with the same number of comparison (s). The general case, where this number varies from group to group, will be treated in Cap. VII.

Now let

$$(1) \qquad m_1, m_2, m_3, \ldots, m_N$$

be the given statistical series whose characteristics are to be determined. The arithmetic mean is defined by the formula

$$(2) \qquad M = \frac{m_1 + m_2 + m_3 + \ldots + m_N}{N}.$$

5 It may seem superfluous to spend many words on so well-known a concept as the arithmetic mean, at least on its numerical computation. However I must exactly in this connexion turn the attention to several points.

The formula (2) can obviously be written in the form

$$(3) \quad M = \frac{m_1 - M_0 + m_2 - M_0 + m_3 - M_0 + \ldots + m_N - M_0}{N} + M_0$$
$$= b + M_0{}^{2}),$$

where M_0 designates any arbitrary number. This formula can be used to simplify the computation of the mean. The number M_0, which it is useful to take not too far from M, is called the provisional mean. Although it plays no indispensible role in the computation of the arithmetic mean, it is of great practical significance in deriving the higher characteristics. It is therefore desirable to become early acquainted with its use.

In Table 3 I have given a practical example of the provisional mean. Computing the mean of the numbers in the second column, we have M = 6196/24 = 258.2 .

1. In general I shall use such names as have nearly the same form in different languages, and most preferably names borrowed from the Latin language.

<I have not kept the name *medium* from fear that English readers will confuse it with the statistical concept *median*. It should also be noted that the name *dispersion* is frequently given to σ^2. --Tr.>

2. Here, as in the sequel, I designate the quantity $M - M_0$ by b.

Table 3.

Number of boys per 500 births in different provinces of Sweden in May 1883.

$s = 500,\ N = 24,\ M_0 = 250.$

k	m_k	$m_k - M_0$		$(m_k - M_0)^2$
1	244		-6	36
2	243		-7	49
3	231		-19	361
4	275	$+25$		625
5	264	$+14$		196
6	256	$+6$		36
7	257	$+7$		49
8	250	$+0$		0
9	240		-10	100
10	266	$+16$		256
11	271	$+21$		441
12	259	$+9$		81
13	256	$+6$		36
14	246		-4	16
15	263	$+13$		169
16	246		-4	16
17	267	$+17$		289
18	280	$+30$		900
19	259	$+9$		81
20	244		-6	36
21	252	$+2$		4
22	282	$+32$		1024
23	261	$+11$		121
24	284	$+34$		1156
	6196	$+252$	-56	6078

Employing the provisional mean $M_0 = 250$, we have

$$b = (252 - 56) : 24 = 8.2$$

and therefore

$$M = 8.2 + 250 = 258.2$$

as before.

The last column of the table is used in computing the dispersion (compare the next chapter).

With respect to the application of the provisional mean we remark that the value of M which we get is the same, whether or

5 not the provisional mean is applied. The computation of all other characteristics, however, is not only more easy but more exact when M_0 is used. For in ordinary calculations M must be rounded to a small number of decimals, preventing application of the exact value of M. This cannot happen with M_0 and thus the entire computation becomes simpler and the result more exact with less work.

6 Division into classes.

If the number of elements in the statistical series is very large, direct computation of the mean by formula (2) or (3) is very laborious. It is then useful to arrange the elements in classes, so that the elements of approximately the same size are taken together in the same class. For this the class interval (w) and the class limits must be properly chosen; their meaning is clear from the following numerical example.

The number of newborn boys in each month for each province (*i.e.* 'län') of Sweden (with the exception of Gotland) was taken from the official statistics of Sweden for the years 1883 and 1890, and uniformly reduced to the basis of boys per 500 births per province per month. The 576 elements obtained in this way, which varied between the lowest value 202 (Jämtland, February 1890) and the highest value 300 (Wärmland, September 1890), were arranged in classes with a class interval $w = 5$.[3]

The class limits were so chosen that all elements between 200 and 204 were put into one class, all elements between 205 and 209 into one class, etc. Thus was obtained the following table:

The first column gives the class limits, the second the corresponding centre of each class. These columns may generally be omitted, since they are sufficiently described by the values M and w above the table. The fourth column gives the frequency, *i.e.* the number of elements of the statistical series within each class. We find that the number of boys in 500 newborn lies only once between the limits 200 and 204 , but 18 times between 235 and 239 , 108 times between 255 and 259 , etc. The frequency is generally designated $F(x)$, so that $F(x)$ designates the frequency in a class with the mark x.

The remaining columns of Table 4 are now easily understood. To compute the mean a provisional mean is chosen. We here take the class 255-259 , in which falls the largest number of elements. M_0, the centre of this class, is 257 . The other classes now become -5 , -10 , -15 , etc., in the negative and +5 , +10 , +15 , etc., in the positive direction. It is best to take the class interval as a unit; the classes are then designated by the numbers given. These numbers are called class-marks in the table and designated x.

3. The class interval should be chosen as a convenient number, such that the dispersion is not less than $4w$, nor the range than $20w$.

Table 4.

Number of boys per 500 births in Sweden for each month of the years 1883 and 1890.

$s = 500$, $N = 576$, $M_0 = 257$, $w = 5$.

Class			Frequency	$x\,F(x)$	
Limits	centre	mark $= x$	$= F(x)$	pos.	neg.
200—204	202	−11	1		−11
205—209	207	−10	0		0
210—214	212	−9	0		0
215—219	217	−8	1		−8
220—224	222	−7	2		−14
225—229	227	−6	5		−30
230—234	232	−5	13		−65
235—239	237	−4	18		−72
240—244	242	−3	47		−141
245—249	247	−2	60		−120
250—254	252	−1	81		−81
255—259	257	0	108		0
260—264	262	+1	91	+91	
265—269	267	+2	60	+120	
270—274	272	+3	44	+132	
275—279	277	+4	22	+88	
280—284	282	+5	16	+80	
285—289	287	+6	6	+36	
290—294	292	+7	0	0	
295—299	297	+8	0	0	
300—304	302	+9	1	+9	
			576	+556	−542

The computation is now very simple. The products $xF(x)$ are easily found. Their sum, divided by 576 (= N), gives the distance from the provisional to the arithmetic mean. We have then

$$b = w\,(556 - 542) : 576 = +0.0243\,w = +0.122$$

and

$$M = 257 + b = 257.122,$$

where the third decimal may be omitted.[4]

4. Compare the chapter on mean error.

Cap. III. The Dispersion.

The dispersion is designated σ and is defined by the formula

7 (1) $$\sigma^2 = \frac{(m_1 - M)^2 + (m_2 - M)^2 + \ldots + (m_N - M)^2}{N}$$

It can be computed directly by this formula or better by using the provisional mean, in which case the formula reads:

(2) $$\sigma^2 = \frac{(m_1 - M_0)^2 + (m_2 - M_0)^2 + \ldots + (m_N - M_0)^2}{N} - b^2,$$

where, as usual, $b = M - M_0$

Taking the example in Table 3 , we have

$$\sigma^2 = 6078 : 24 - (8.2)^2 = 186.0$$

and therefore

$$\sigma = 13.64.$$

If the elements are arranged in classes, the computation--with
8 check--runs as follows. I take the same example as in Table 4.

Table 5.

Number of boys per 500 births.

$s = 500,\ N = 576,\ M_0 = 257,\ w = 5.$

$(x+1)^2$	x	$F(x)$	$x\,F(x)$	$x^2\,F(x)$	$(x+1)^2\,F(x)$
100	−11	1	−11	+121	100
81	−10	0	0	0	0
64	−9	0	0	0	0
49	−8	1	−8	+64	49
36	−7	2	−14	+98	72
25	−6	5	−30	+180	125
16	−5	13	−65	+325	208
9	−4	18	−72	+288	162
4	−3	47	−141	+423	188
1	−2	60	−120	+240	60
0	−1	81	−81	+81	0
1	0	108	0	0	108
4	+1	91	+91	+91	364
9	+2	60	+120	+240	540
16	+3	44	+132	+396	704
25	+4	22	+88	+352	550
36	+5	16	+80	+400	576
49	+6	6	+36	+216	294
64	+7	0	0	0	0
81	+8	0	0	0	0
100	+9	1	+9	+81	100
		576	+14	+3596	+4200

The expression (2) for the dispersion takes now the following form:

$$\sigma^2 = w^2 \left\{ \frac{\Sigma x^2 F(x)}{N} - b^2 \right\},$$

where b is expressed in class interval units and $\Sigma x^2 F(x)$ signifies the sum of all numbers in the fifth column of Table 5 .

From the table we have

$$\Sigma x^2 F(x) = 3596,$$

and in Art. 6 we found $b = +0.024\ w$. Therefore

$$\sigma^2 = w^2 \{ 3596 : 576 - (0.024)^2 \} = w^2 6.242$$

and

$$\sigma = w\, 2.498 = 12.49.$$

The entire computation is carried out in these lines. A direct calculation by formula (1), without using the provisional mean, would be the work of a whole day.

The computation is easily checked with the help of the numbers in the first and last columns of Table 5 .

For, as is easy to see,

$$\Sigma (x+1)^2 F(x) = \Sigma x^2 F(x) + 2\,\Sigma x\, F(x) + \Sigma F(x).$$

The numbers in the last line of Table 5 give

$$\begin{aligned} \Sigma x^2 F(x) &= +3596 \\ 2\,\Sigma x\ F(x) &= +\ \ \ 28 \\ \Sigma\ \ F(x) &= +\ \ 576 \\ \hline &\ +4200 = \Sigma (x+1)^2 F(x). \end{aligned}$$

9 Presupposing that the higher characteristics--and in particular the skewness and the excess--are small, the dispersion enables us to compute the distribution of the elements of the statistical series into the several classes very easily by means of the Gaussian law of error. I shall return to this point in the chapter on frequency curves and will only mention here one simple property of the dispersion: the number of elements of the series between the limits $M-\sigma$ and $M+\sigma$ is about 2/3 of the whole number N. In the example at hand the dispersion is about 2½ class intervals. The combined number of elements in the classes with the marks -2 , -1 , 0 , +1 , and +2 is 400 in the table, and 2/3 of 576 is 384 , so that in this example somewhat more than 2/3 of the total number of elements lie between the limits $M+\sigma$ and $M-\sigma$.

9 Taking the example in Table 3 , where M = 258.2 , σ = 13.6 , we have $M+\sigma$ = 271.8 and $M-\sigma$ = 244.6 . The number of elements in Table 3 between these limits is 16 and exactly equals 2/3 of the total number of elements (24).

For this reason the dispersion is an excellent measure of the variance[1] about the mean.

On this property is founded the use of the dispersion to judge the uncertainty in determination of quantities from statistical observations. We shall treat this question in the next chapter.

10 The dispersion can also be used to determine the limits between which the elements of a statistical series lie on the average. The theoretical results in this connexion are collected in the following table.

Table 6.

The limits, either side the mean, beyond which an average of one element (of a statistical series with N elements) will fall.

N	Limits	N	Limits
10	$\pm 1.65\,\sigma$	300	$\pm 2.94\,\sigma$
20	$1.96\,\sigma$	400	$3.0\,\sigma$
30	$2.18\,\sigma$	500	$3.0\,\sigma$
40	$2.24\,\sigma$	600	$3.1\,\sigma$
50	$2.33\,\sigma$	700	$3.2\,\sigma$
60	$2.39\,\sigma$	800	$3.2\,\sigma$
70	$2.45\,\sigma$	900	$3.2\,\sigma$
80	$2.50\,\sigma$	1 000	$3.3\,\sigma$
90	$2.54\,\sigma$	10 000	$3.9\,\sigma$
100	$2.58\,\sigma$	100 000	$\pm 4.4\,\sigma$
200	$\pm 2.81\,\sigma$		

Taking the example in Table 3 (N = 24 , σ = 13.6 , M = 258.2), we have, from Table 6 , the limits ±2.03 σ = ±27.6 . We may then expect that a single element of the series is smaller than 258.2 - 27.6 = 230.6 or larger than 258.2 + 27.6 = 285.8 . In fact there is no element without these limits, but two which lie near the limits. The lowest element is 231 (No. 3), the highest 284 (No. 24); in good correspondence with theory.

In Table 4 (N = 576 , σ = 12.5 , M = 257.1) we have the limits M - 3.1 σ = 218.4 and M + 3.1 σ = 295.8 . Without these limits we have among the 576 elements of Table 4 three elements. This is

1. The dispersion has many different names in the literature. The expression of GAUSS is *mittlere Abweichung*, PEARSON applies the name *standard deviation*, some German mathematicians say *Streuung*. The name dispersion has the advantage of being understandable in any language. <In English the term *mean deviation* is ill advised, as suggesting rather the *average deviation* (ϑ). The name *Streuung*, and its equivalent, variance, have been used for σ^2. --Tr.>

properly two too many. But firstly Table 6 is valid only on the average and secondly we do not yet know whether the higher characteristics may be neglected. However the smallest element (202 boys in 500 births, Jamtland, February 1890), which differs from the mean by more than four times the dispersion, appears to be erroneous. Therefore I applied to Dr WIDELL, director of the central bureau of statistics, who very kindly had the original data for this element reread. A few small errors were indeed found, which, however, do not materially affect the result.[2]

11 The average deviation (θ). A convenient measure of variance is furnished by the average deviation. It is designated θ and defined by the formula

$$\theta = \frac{|m_1 - M| + |m_2 - M| + |m_3 - M| + \ldots + |m_N - M|}{N}$$

where m_1-M designates the difference between m_1 and M, taken positive (even if m_1 is less than M). If we take, *ex. gr.*, the series of Table 3 , the computation of θ runs as in Table 7 below. Take the differences m-M (from Art. 5 , M = 258.2) and add them, ignoring sign. The sum of the numbers in the second column is 266 and therefore

$$\theta = 266 : 24 = 11.08.$$

From the average deviation, the dispersion can be approximately computed by the formula

$$\sigma = \sqrt{\frac{\pi}{2}}\,\theta = 1.2533\,\theta. \tag{4}$$

In the present example, computing σ from formula (4), we have σ = 13.9 , whereas by direct computation by formula (2) we found σ = 13.6 .

Table 7.

$s = 500,\ N = 24$

m	$m - 258{,}2$
244	14.2
243	15.2
231	27.2
275	16.8
264	5.8
256	2.2
257	1.2
250	8.2
240	18.2
266	7.8
271	12.8
259	0.8
256	2.2
246	12.2
263	4.8
246	12.2
267	8.8
280	21.8
259	0.8
244	14.2
252	6.2
282	23.8
261	2.8
284	25.8

2. It must be remembered in this connexion that the series of Table 4 was not directly observed, but that the numbers were reduced to the basis of 500 births per province. Compare in this respect the remarks of Chapter VII.

Cap. IV. On mean Error.

12 The mean error of a quantity x is denoted by $\varepsilon(x)$ (or briefly ε). This means that on the average, in a statistical computation of x, the value of x is found two times out of three to lie within, and once out of three to lie without, the limits $M-\varepsilon$ and $M+\varepsilon$. Here M signifies the mean of all values of x.

13 The mean error of an element in a statistical series is equal, then, to the dispersion of the series; or expressed in a formula,

$$\varepsilon(m) = \sigma. \tag{1}$$

The mean error of an element of the series in Table 5 is equal, by Art. 8 , to 12.5 . This can be expressed (not strictly correctly) in words thus: choosing at random (from those in Table 5) 500 newborn children, in two cases out of three the number of boys in these 500 will lie between the limits 257 - 12.5 = 244.5 and 257 + 12.5 = 269.5 .

14 The mean error of the mean is given by the formula

$$\varepsilon(M) = \frac{\sigma}{\sqrt{N}} \tag{2}$$

from which it is seen that the mean error of the mean varies inversely as the square root of the number of elements.

By formula (2) the mean error of the mean of the series in Table 3 is $13.64/\sqrt{24} = 2.78$; that of that in Table 5 is $12.49/\sqrt{576} = 0.520$.

It is customary[1] in giving a statistical value to add its mean error with the sign $\pm$. Thus the mean number of boys in 500 births from Table 3 is written

$$258.2 \pm 2.78$$

and from Table 5

$$257.12 \pm 0.520.$$

15 The mean error of the dispersion (σ) is given by the formula

$$\varepsilon(\sigma) = \frac{\sigma}{\sqrt{2N}} \tag{3}$$

Applying this formula to the examples of Tables 3 and 5 , the dispersion with mean error added from Table 3 is written

1. <Among English writers, the sign $\pm$ is frequently used with the probable error, a quantity defined by P.E. = $0.67449\,\epsilon$. --Tr.>

$$\sigma = 13.64 \pm 1.97$$

and from Table 5

$$\sigma = 12.49 \pm 0.367.$$

Notice that the determination of σ (and also of M) from Table 5 is significantly more certain than from Table 3 . This is caused by the different number of elements. The difference between the two values is smaller than the mean error. Compare Art. 17 .

16 The mean error of the average deviation (θ) is given by

$$(4) \qquad \varepsilon(\theta) = \sqrt{\pi - 2}\,\frac{\theta}{\sqrt{2N}} = 1.0685\,\frac{\theta}{\sqrt{2N}}$$

In the example of Art. 11 we have θ = 11.08 and N = 24 , and therefore, with the mean error appended,

$$\theta = 11.08 \pm 1.71.$$

17 The mean error of the sum of two observed quantities a and b is given by the formula

$$(5) \qquad \varepsilon(a + b) = \sqrt{\varepsilon^2(a) + \varepsilon^2(b)}$$

The mean error of the difference between a and b is obtained from the formula

$$(6) \qquad \varepsilon(a - b) = \sqrt{\varepsilon^2(a) + \varepsilon^2(b)}$$

The mean error of the sum is thus equal to the mean error of the difference.

The mean error of a multiple ka of a is

$$(7) \qquad \varepsilon(ka) = k\,\varepsilon(a).$$

From (6) and (7) follows the more general formula

$$(7^*) \qquad \varepsilon(k_1 a + k_2 b) = \sqrt{k_1^2 \varepsilon^2(a) + k_2^2 \varepsilon^2(b)}.$$

As example for (7) let us compute the mean error of the probability of birth of a boy. This is obtained from Table 3 or 5 by dividing by the number of comparison (s), or by multiplying by $1/s$. Taking the larger series (Table 5), we find by Art. 14 the value 0.51424 ± 0.00104 for the probability of birth of a boy.

18 The number of decimals in the numerical value of a quantity is best determined from its mean error. It is convenient to proceed according to the following rules:

18 1. Express the mean error with three (possibly two) significant figures;

2. give the quantity itself with the same accuracy (the same number of decimals) as its mean error or possibly with one decimal fewer.

All numerical values in this chapter are given in accordance with these rules.

19 If on two different occasions the values a and b have been found for a characteristic of a statistical series, the correspondence between the two values is called good if the difference between a and b is numerically smaller than the mean error of this difference, given by formula (6). The correspondence is satisfactory, if the difference $a-b$ remains less than twice (exceptionally three times) its mean error. If the difference $a-b$ were to increase over three times (possibly twice) its mean error, the correspondence must be considered less good. In general there is then occasion to hope that a plausible explanation for this deviation can be discovered. Only once in 27 000 cases does a value of the difference $a-b$ accidentally occur that is greater than four times its mean error.

Cap. V. Bernoulli's Theorem.

The simplest statistical Series.

20 We assume that s cards are successively drawn from a pack of m black and n red cards, the card drawn being replaced in the pack after each drawing. In these s drawings are drawn say m_1 black cards in all. The experiment is repeated N times; let

$$(1) \qquad m_1, m_2, m_3, \ldots, m_N$$

be the number of black cards obtained in these N trials (each trial encompassing s simple drawings). Then the numbers (1) form what I call a Bernoulli series or the simplest statistical series.

21 The characteristics of this series can be calculated from Bernoulli's theorem [5]. Let p designate the ratio of the number of black cards to the total number of cards and q the ratio of the number of red cards to the total number of cards, so that

$$(2) \qquad p = \frac{m}{m+n}, \quad q = \frac{n}{m+n}$$

and therefore $p+q = 1$. The numbers p and q are called respectively the probabilities for the drawing of a black and of a red card.

Bernoulli's theorem states that the mean of (1), which we shall call the Bernoulli mean and designate M_B, is obtained from the formula

(3) $$M_B = sp$$

and that the dispersion of (1), which we shall call the BERNOULLI dispersion and designate σ_B, is given by the formula

(4) $$\sigma_B = \sqrt{spq}.$$

22 The correctness of formulae (3) and (4) can easily be tested by experiment, although it is rather laborious to collect material in sufficient quantity. I have succeeded in obtaining, by the kind collaboration of some friends, a collection of 12 600 of these drawings from an ordinary pack of 52 cards, 10 000 of which are designed to illustrate BERNOULLI's theorem. The others are applied in the following chapters.

These drawings may be arranged in groups with an arbitrary number of comparison (s), and I shall give here the results for three values of s: $s = 500$, $s = 50$, and $s = 10$, and compare the mean and dispersion of these series with their values (3) and (4) according to BERNOULLI.

23 We begin with the case $s = 500$. 20 groups can be formed in all. Since the computation of the characteristics has here a very simple form, I give it at length. The provisional mean is conveniently taken at 250. Table 8 now gives

$$b = (85 - 152) : 20 = -3.35$$

and therefore

$$M = 246.65 \pm 3.09.$$

The mean error is computed by formula (2) of the preceding chapter.

Moreover we have

$$\sigma^2 = 4045 : 20 - b^2 = 191.03,$$

so that, with mean error added,

$$\sigma = 13.82 \pm 2.18.$$

Table 8.

Number (m) of black cards in 500 drawings.

$s = 500,\ N = 20,\ M_0 = 250.$

m	$m - M_0$		$(m - M_0)^2$
252	+ 2		4
235		− 15	225
248		− 2	4
271	+21		441
260	+10		100
246		− 4	16
228		− 22	484
229		− 21	441
234		− 16	256
250	0		0
271	+21		441
234		− 16	256
258	+ 8		64
233		− 17	289
273	+23		529
244		− 6	36
249		− 1	1
241		− 9	81
231		− 19	361
246		− 4	16
	+85	−152	+4045

23 We will now compare these values with those given by the formulae of BERNOULLI. Because in all these trials $p = q = \frac{1}{2}$, we have from (3) and (4)

$$M_B = 500 \times {}^1/_2 = 250,$$

$$\sigma_B = \sqrt{500 \times {}^1/_2 \times {}^1/_2} = 11.18.$$

It is seen that M is somewhat too small and σ somewhat too large. The correspondence, however, in the terminology of Art. 19 , is satisfactory, since the difference is in no case more than 1½ times the mean error.

24 We next collect the trials in groups of 50 drawings, i.e. $s = 50$. Obviously 10 000 drawings divide into 200 of these groups. It is now advantageous to arrange the elements in classes. The result of the experiment is to be seen in Table 9 : once were 14 black cards obtained in 50 drawings, no times 15 black cards, twice 16 cards, etc.

The computation of the characteristics proceeded as per the instructions of Art. 8 and the result, with mean error added, was

$$M = 24.685 \pm 0.248,$$

$$\sigma = 3.510 \pm 0.176,$$

whereas formulae (3) and (4) give:

$$M_B = 25,$$

$$\sigma_B = 3.536.$$

The obtained dispersion lies within the limits of its mean error, while the mean is here again somewhat too small.[1] The difference, however, is again less than 1.5 ε.

Table 9.

Number (m) of black cards in 50 drawings.

$s = 50,\ N = 200,$

$M_0 = 25,\ w = 1.$

m	x	$F(x)$
14	−11	1
15	−10	0
16	− 9	2
17	− 8	2
18	− 7	4
19	− 6	8
20	− 5	6
21	− 4	15
22	− 3	13
23	− 2	15
24	− 1	34
25	0	14
26	+ 1	21
27	+ 2	28
28	+ 3	14
29	+ 4	10
30	+ 5	5
31	+ 6	5
32	+ 7	3
33	+ 8	2
		200

1. The mean computed will deviate in the same direction from M_0, no matter how the experiments are grouped.

25 If the trials are arranged in groups of 10 drawings, the following result is obtained. Because no fewer than 1000 elements are obtained here, division into classes is necessary. I give here the complete computation, which is not laborious.

Table 10.

Number (m) of black cards in 10 drawings.

$s = 10,\ N = 1000,\ M_0 = 5,\ w = 1.$

$(x+1)^2$	m	x	$F(x)$	$x\,F(x)$	$x^2\,F(x)$	$(x+1)^2\,F(x)$
16	0	-5	3	-15	$+75$	$+48$
9	1	-4	10	-40	$+160$	$+90$
4	2	-3	43	-129	$+387$	$+172$
1	3	-2	116	-232	$+464$	$+116$
0	4	-1	221	-221	$+221$	0
1	5	0	247	0	0	$+247$
4	6	$+1$	202	$+202$	$+202$	$+808$
9	7	$+2$	115	$+230$	$+460$	$+1035$
16	8	$+3$	34	$+102$	$+306$	$+544$
25	9	$+4$	9	$+36$	$+144$	$+225$
36	10	$+5$	0	0	0	0
			1000	-67	$+2419$	$+3285$

Check:

$$
\begin{aligned}
\Sigma x^2 F(x) &= +2419 \\
2\Sigma x\ F(x) &= -\ 134 \\
\Sigma\ \ F(x) &= +1000 \\
\hline
&\ \ +3285
\end{aligned}
$$

Hence we have

$$b = -67 : 1000 = -0.067,$$

and so, with mean error, according to Art. 14 ,

$$M = 5 - 0.067 = 4.933 \pm 0.050$$

$$\sigma^2 = 2.419 : 1000 - b^2 = 2.415,$$

and, since

$$\sigma = 1.554 \pm 0.035.$$

For the BERNOULLI mean and the BERNOULLI dispersion we have from (3) and (4) the values

25 $$M_B = 5.$$

$$\sigma_B = 1.581.$$

The agreement is as good as that of the previous example.

Cap. VI. The Theorems of POISSON and LEXIS.

26 POISSON's theorem. We now assume that in our trials--each s drawings from a pack of cards--the ratio between the numbers of black and of red cards varies from drawing to drawing. Let p_1 be the probability of getting a black card in the first drawing, p_2 the corresponding probability in the second, p_3 in the third, etc. Likewise let q_1, q_2, q_3 be the corresponding probabilities of getting a red card.

The trial is repeated N times, changing the composition of the pack in the same way within each trial of s drawings.

If m_1, m_2, m_3, ..., m_N are the number of black cards obtained in each of these trials, then I say that the numbers (the elements)

$$m_1, m_2, m_3, \ldots, m_N \tag{1}$$

form a POISSON series.

The mean (M_P) and the dispersion (σ_P) of this series were first computed by POISSON and have the following values:

$$M_P = p_1 + p_2 + p_3 + \ldots + p_s = \Sigma p_k, \tag{2}$$

$$\sigma^2_P = p_1 q_1 + p_2 q_2 + p_3 q_3 + \ldots + p_s q_s = \Sigma p_k q_k. \tag{3}$$

We will designate the arithmetic mean of the s values of p by p_0, that of the q by q_0, so that

$$\begin{aligned} p_0 &= \frac{p_1 + p_2 + p_3 + \ldots + p_s}{s}, \\ q_0 &= \frac{q_1 + q_2 + q_3 + \ldots + q_s}{s}. \end{aligned} \tag{4}$$

If all drawings had been performed with the constant probabilities p_0 and q_0, BERNOULLI's theorem would yield

$$M_B = s\,p_0, \tag{5}$$

$$\sigma_B^2 = s\,p_0\,q_0 \tag{6}$$

$$M_P = M_B. \tag{7}$$

26 Hence follows: if the s drawings had been performed all with constant probability of black card p_0, instead of with the varying probabilities p_1, p_2, p_3, ..., p_s, the mean would have been the same as in the POISSON series.

For the dispersion, however, is found, after a short calculation, the formula

$$\sigma_P^2 = \sigma_B^2 - \Sigma (p_k - p_0)^2, \tag{8}$$

so that the dispersion of the POISSON series is always smaller than the BERNOULLI dispersion corresponding to the probability p_0.

It can however be proved that the difference between these values of dispersion is generally insignificant.

27 I have chosen the following experiment to illustrate POISSON's theorems (7) and (8). From a pack of cards with 13 cards of each suit one card was drawn at random and the colour noted. Before the next drawing a spade was removed and replaced by a heart from another pack, so that the pack consisted of 12 spades, 13 clubs, 13 diamonds and 14 hearts. From this pack a card was now drawn and the colour noted. Then again a spade was removed and replaced by a heart, whereupon a new drawing was made. This procedure was continued until all the spades had been removed and replaced by hearts. Then this operation was continued with the clubs, which, after drawings, were replaced one by one by diamonds. In this manner 27 (= s) simple drawings were obtained.

These 27 drawings together form one trial. In all 100 (= N) such trials were made, consisting each of 27 (= s) drawings according to the plan above. The result and the computation of the characteristics proceeds from the following table.

We obtain hence b = +0.16 and, with mean error,

$$M = 7.16 \pm 0.194$$

$$\sigma^2 = 3.78 - (0.16)^2 = 3.754,$$

so that

$$\sigma = 1.937 \pm 0.138.$$

According to (2) and (3) the values from the POISSON theory are

$$M_P = 6.75$$

$$\sigma_P = 2.111.$$

The agreement of the mean is not particularly good; that of the dispersion is satisfactory.

27

Table 11.

Poisson series. Number (m) of black cards in 27 (s) drawings.

The probability varies *within* each trial (from one drawing to the next), but not from one trial to the next.

$$s = 27,\ N = 100,\ M_0 = 7,\ w = 1.$$

$(x+1)^2$	m	x	$F(x)$	$xF(x)$	$x^2F(x)$	$(x+1)^2F(x)$	Check
9	3	−4	2	−8	+32	18	
4	4	−3	6	−18	+54	24	
1	5	−2	14	−28	+56	14	
0	6	−1	14	−14	+14	0	+378
1	7	0	22	0	0	22	+32
4	8	+1	17	+17	+17	68	+100
9	9	+2	14	+28	+56	126	+510
16	10	+3	8	+24	+72	128	
25	11	+4	1	+4	+16	25	
36	12	+5	1	+5	+25	36	
49	13	+6	1	+6	+36	49	
			100	+16	+378	510	

The arithmetic mean of the probabilities of a black card is ¼ (= p_0). If all drawings had been performed with this constant probability, the result would have been, according to (5) and (6),

$$M_B = 6.75 = 27 \times {}^1/_4,$$

$$\sigma_B = 2.25 = \sqrt{27 \times {}^1/_4 \times {}^3/_4},$$

which numbers obey POISSON's theorems ($M_P = M_B$, $\sigma_P < \sigma_B$).

28 LEXIS' theorem. Much more important than the theorem of POISSON, however, is a theorem proved by LEXIS.

We will perform N trials with drawings from a pack of cards in the following manner. In the first trial let the probability of getting a black card be p_1. Perform s drawings and get m_1 black cards. In the second trial the composition of the pack is different. The probability of getting a black card is now p_2. Make s drawings again and get m_2 black cards. In this way N trials are made, where the composition of the pack changes from trial to trial, but remains constant among the s drawings which make up a trial. In this manner is produced a statistical series

$$m_1, m_2, m_3, \ldots, m_N,$$

which I call a LEXIS series.

If M_L and σ_L are the mean and the dispersion of this series,

$$M_L = s p_0, \tag{9}$$

where

$$p_0 = \frac{p_1 + p_2 + p_3 + \cdots + p_N}{N} \tag{10}$$

and

$$\sigma_L^2 = s p_0 q_0 + (s^2 - s)\,\sigma_p^2, \tag{11}$$

where

$$\sigma_p^2 = \frac{(p_1 - p_0)^2 + (p_2 - p_0)^2 + (p_3 - p_0)^2 + \cdots + (p_N - p_0)^2}{N} \tag{12}$$

29 Some important conclusions follow from these formulae. I present here only the following of these:

1. The mean of a LEXIS series is the same as that of a BERNOULLI series with the constant probability p_0. $M_L = M_B$.
2. The dispersion of a LEXIS series is greater than the dispersion of the said BERNOULLI series. $\sigma_L > \sigma_B$, where $\sigma_B = \sqrt{s p_0 q_0}$.
3. The ratio σ_L/σ_B is little larger than 1 for small s, but goes to infinity as the square root of s.

The third of these consequences (or more exactly its first part) was first enunciated by BORTKIEWICZ and was named by him the law of small numbers.

The ratio σ_L/σ_B--which I call the LEXIS ratio and designate by L--plays a significant role in practical statistics. I take the opportunity of giving examples of this in the sequel. LEXIS says that a statistical series has supernormal dispersion when $L > 1$, normal, when $L = 1$, and subnormal, when $L < 1$. The LEXIS series defined above always has, by (11) supernormal dispersion, whereas a POISSON series has, by (8), subnormal dispersion. We shall find that most series in applied statistics have supernormal dispersion.[1]

30 Before treating, in the next chapter, some examples from applied statistics, I shall illustrate the theorem of LEXIS by an experimental series as I have illustrated POISSON's theorem.

10 trials, each consisting of 10 simple drawings, were made

1. As an example of series with subnormal dispersion, the number of twin births in Sweden with the total births as number of comparison may be adduced.

30 with an ordinary pack of cards, and the number of black cards got in each trial noted. Then 10 new trials were made with a pack consisting of 25 black and 27 red cards. Then 10 trials with a pack of 24 black and 28 red cards, etc. Of the 270 trials which were performed in this way (until the pack consisted only of red cards) I take the 100 first, which give the following result.

Table 12.

Lexis series. Number of black cards in 10 drawings.

The probability is constant within each trial,
but varies from one trial to the next.

$$s = 10, \quad N = 100, \quad M_0 = 4.$$

$(x+1)^2$	m	x	$F(x)$	$xF(x)$	$x^2F(x)$	$(x+1)^2F(x)$	Check
4	1	−3	4	−12	+36	+16	+294
1	2	−2	9	−18	+36	+9	+76
0	3	−1	19	−19	+19	+0	+100
1	4	0	21	0	0	+21	+470
4	5	+1	23	+23	+23	+92	
9	6	+2	10	+20	+40	+90	
16	7	+3	12	+36	+108	+192	
25	8	+4	2	+8	+32	+50	
			100	+38	+294	+470	

Hence we have $b = +0.38$, and therefore

$$M = 4.38 \pm 0.167,$$

$$\sigma^2 = 294:200 - b^2 = +2.796,$$

so that

$$\sigma = +1.672 \pm 0.118.$$

The mean probability (p_0) for all trials was

$$p_0 = 21.50:52 = 4.4135,$$

from which we have

$$M_B = sp_0 = 4.135,$$

$$\sigma_B = \sqrt{sp_0 q_0} = 1.557.$$

Computing the values according to LEXIS from (9) and (11)--computing the dispersion is somewhat lengthy--we have

$$M_L = 4.135,$$

$$\sigma_L = 1.643.$$

The agreement between the observed values and those of LEXIS is satisfactory. Comparing the latter with the values according to BERNOULLI, we find theorems 1. and 2. above confirmed.

The LEXIS ratio (L) has the value 1.06 . The series has therefore a very slightly supernormal dispersion.

Arranging all the data into 27 trials, of 100 drawings each, a series is obtained for which L = 3.82 .

The dispersion is here supernormal in significantly high degree, in agreement with theorem 3. of Art. 29 .

In closing it must be noted that by a combination of the theorems of POISSON and LEXIS it is possible theoretically to reproduce any arbitrary homograde statistical series.

Cap. VII. The observed statistical Series.

31 Take as given a series of numbers which express the number of newborn children in a certain country in various years. Let these numbers be $m_1, m_2, m_3, \ldots, m_N$. Suppose moreover for simplicity's sake that the number of inhabitants has remained constant during the period observed. Then the ratios m_1/s, m_2/s, m_3/s, ..., m_N/s may be considered as the observed probabilities of the birth of a child in the years in question. This identification of a statistical quantity with a mathematical probability is only an analogy. Very possibly it has little connexion with the observed statistical phaenomena; but closer investigation shews the great significance in statistics of such a train of thought.

The next course is to consider the numbers of the statistical series $m_1, m_2, m_3, \ldots, m_N$ as analogous to the number of 'favourable' cases in N trials, where each trial consists of s drawings (from a pack of cards); all these performed with the constant probability p_0, approximately expressible by the arithmetic mean of the empirical probabilities m_1/s, m_2/s, ..., m_N/s.

31 The dispersion and the other characteristics of the observed statistical series are then given by BERNOULLI's theorem. The founders of mathematical statistics regarded the identification of an observed statistical series with a BERNOULLI series almost axiomatically evident. LAPLACE likewise was led by this identification to incorrect conclusions, *inter al.* with respect to the census of population in France which he instigated.

Only after the appearance of LEXIS was the untenability of this way of thinking clearly realised, causing a clearer insight into the nature of statistical series.

32 I shall study in this chapter some examples of statistical series, which have all been taken from the official statistics of Sweden. All figures given here have been reduced, unless otherwise stated, to a population of 5 million men as basis, so that the number of comparison s = 5 000 000 . This reduction requires a small correction which will be treated in the next chapter. It is, however, insignificant in the examples at hand, and we shall assume the figures given as directly observed for a population of 5 000 000 men.

We treat first the question of the number of new-born children in Sweden in a year. In the interest of clarity I give in this case the computation of the mean and dispersion in full detail.

$$b = (53\,190 - 50\,390) : 20 = 140.$$

$$M = M_0 + b = 140\,140.$$

$$\sigma^2 = 654\,401\,400 : 20 - b^2 = 32\,700\,470,$$

so that

$$\sigma = 5718.$$

The empirical probability of birth (p_0) is

$$p_0 = M : s = 0.02803,$$

therefore we have

$$q_0 = 1 - p_0 = 0.97197$$

and the BERNOULLI dispersion is

$$\sigma_B = \sqrt{s\,p_0\,q_0} = 369.0.$$

The actually observed dispersion (5718) is therefore much larger than the BERNOULLI dispersion. The series of births is significantly supernormal. The LEXIS ratio (L) has the value

$$L = 5718 : 369.0 = 15.50.$$

Table 13.

Number (m) of births in Sweden.

(rounded to tens).

$s = 5\,000\,000,\ N = 20,\ M_0 = 140\,000.$

Year	m	$m - M_0$		$(m - M_0)^2$
1881	145 230	+ 5 230		27 352 900
1882	146 640	+ 6 640		44 089 600
1883	144 320	+ 4 320		18 662 400
1884	149 360	+ 9 360		87 609 600
1885	146 600	+ 6 600		43 560 000
1886	148 270	+ 8 270		68 392 900
1887	148 020	+ 8 020		64 320 400
1888	143 680	+ 3 680		13 542 400
1889	138 300		— 1 700	2 890 000
1890	139 600		— 400	160 000
1891	141 070	+ 1 070		1 144 900
1892	134 830		— 5 170	26 728 900
1893	136 540		— 3 460	11 971 600
1894	134 840		— 5 160	26 625 600
1895	136 820		— 3 180	10 112 400
1896	135 330		— 4 670	21 808 900
1897	132 750		— 7 250	52 562 500
1898	134 820		— 5 180	26 832 400
1899	131 320		— 8 680	75 342 400
1900	134 460		— 5 540	30 691 600
		+53 190	—50 390	654 401 400

33 Table 14 gives the numbers of deaths, drownings, suicides, marriages, and divorces in Sweden during the years 1876-1900 .

The values of the mean (M), of the dispersion (σ), of the BERNOULLI dispersion (σ_B), and of the LEXIS ratio (L), are collected in Table 15 .

In all these cases supernormal dispersion is found, the most in the number of deaths, the least--although still significant--in the number of divorces.

34 Dispersion which is not supernormal is met extremely seldom in applied statistics. The classic example for a series in which normal dispersion is ordinarily expected is the number of new-born boys, with the total number of births as number of comparison. We found in Art. 8 from observations in Sweden in the years 1883 and 1890

Table 14.

Series from the official statistics of Sweden.

$s = 5\,000\,000,\ N = 25.$

Year	Deaths	Drownings	Suicides	Marriages	Divorces
1876	97 450	1349	462	35 200	239
1877	92 740	1261	479	34 200	235
1878	89 830	1239	453	32 160	226
1879	84 240	1219	478	31 270	225
1880	90 630	1349	421	31 670	238
1881	88 360	1164	420	30 950	234
1882	86 700	1564	526	31 630	213
1883	86 330	1279	510	31 980	237
1884	87 280	1236	464	32 510	259
1885	88 390	1169	494	33 000	245
1886	82 730	996	601	31 940	240
1887	80 500	1256	541	31 170	246
1888	79 850	1007	595	29 560	265
1889	79 720	1166	561	29 820	251
1890	85 500	1151	637	29 900	309
1891	83 910	1040	638	29 090	287
1892	89 350	950	708	28 440	329
1893	83 980	979	698	28 210	304
1894	81 520	1231	791	28 580	300
1895	75 600	1034	754	29 200	310
1896	77 850	1275	739	29 600	352
1897	76 410	1029	760	30 160	348
1898	75 000	1182	718	30 510	404
1899	87 970	1133	774	31 100	380
1900	83 850	1026	777	30 640	394

Table 15.

	M	σ	σ_B	L
Deaths....................	84 630	5427	288	18.81
Drownings	1 171	140	34	4.10
Suicides	600	124	24	5.08
Marriages................	30 900	1707	175	9.74
Divorces..................	281	55.6	17	3.81

$$s = 500, N = 576,$$

$$M = 257.12, \sigma = 12.49,$$

so that the probability (p_0) of the birth of a boy was

$$p_0 = M : s = 0.51424.$$

The corresponding BERNOULLI dispersion is

$$\sigma_B = 11.18,$$

somewhat smaller than the observed dispersion (12.49). The LEXIS ratio is

$$L = 1.117.$$

The value of L is so close to unity, that it may be doubted whether the deviation is to be considered accidental or not. To decide this, the mean error of L must be known. I shall omit here the formula for this mean error and communicate only the result in this case. L can be written, with mean error added:[1]

$$L = 1.117 \pm 0.033.$$

Since the deviation from unity is somewhat more than three times the mean error, this series can be considered supernormal, although in a slight degree.

We can therefore conclude in accordance with LEXIS' theorem that the determination of the sex of the new-born can not entirely be compared with a lottery, but that outer influences are also at work here.

In order to aid in the investigation of these outer influences, I have arranged the data on birth of boys partly according to time (months and years), and partly according to space (different provinces). It seems that the former division (according to time) gives a normal series--in one case even a subnormal series--, but that a supernormal series is obtained in the division according to provinces. It may be hence concluded that the probability of the birth of a boy does not essentially vary with the time (as far as the present investigation goes), but rather with the place. I conclude that the sex ratio in the number of births may be a racial character--using the word race in the widest sense.

1. Because of the relations treated in the next chapter, however, the value of L must be reduced somewhat.

35 Among the few subnormal statistical series that I have met lies the number of births of twins with the number of single births as number of comparison (in Art. 33 the whole population was used as number of comparison). From the official statistics of Sweden for 1883 I obtained, by comparing the births of twins in the various provinces, $L = 0.78 \pm 0.11$. The mean error of L is too large to exclude the possibility of an accidentally small dispersion in the number of births of twins in this year. If, however, by analysis of more comprehensive data, it were established that the number of births of twins formed a series with subnormal dispersion, the conclusion could be drawn that the fluctuations in the individual probability of birth of twins have a significantly greater influence on the dispersion than the fluctuations which stem from differences in the make-up of the population of the provinces.

It is in the nature of things, as a mathematical analysis of the theorems of POISSON and of LEXIS shews, that statistical series with subnormal dispersion are significantly more difficult to realise than series for which $L > 1$.

36 We found in Art. 32 that for new-born in general the LEXIS ratio had the value 15.5 . Observing instead the number of births of twins, we obtain (with $M = 2015$, $\sigma = 95.2$, $\sigma_B = 44.9$) the value 2.12 for L, a significantly smaller L than for births in general.

This indicates an incompleteness in the LEXIS ratio as a measure of disturbing influences to which a statistical event is exposed. Obviously the factors which operate to disturb the probability of birth operate in nearly the same degree upon the probability of birth of twins; nevertheless the LEXIS ratio was seven times as large in one case as in the other.

As may be seen from the theorem of LEXIS, this incompleteness is caused by the fact that the number L is dependent upon p_0 and that p_0 is smaller for birth of twins than for single birth.

Another defect of L as measure of disturbing forces is that it is dependent upon the number of comparison, varying roughly as $\sqrt{s}$. Taking for example the number of newborn in the city of Lund in the years 1882-1901 , with $s = 16\,000$ inhabitants as number of comparison, we have

$$M = 402, \sigma = 24.9, \sigma_B = 20.0$$

and

$$L = 1.24.$$

The numbers of births in Lund form, therefore, an almost normal series, whereas we have found for Sweden as a whole $L = 15.5$.

Obviously, however, the influence of outside disturbances on

the number of births in Lund is about as strong as in the country generally.

37 For these reasons I have introduced, in place of the LEXIS ratio, or more properly in addition to it, a quantity which is free from these defects and is therefore better suited as measure for the intensity of outside influences. I designate this quantity ρ, putting

$$\rho = \frac{\sqrt{\sigma^2 - \sigma^2_B}}{M} \tag{1}$$

and call 100ρ the coefficient of disturbancy of the statistical series. Its value--with mean error--for some of the series here given is:

Table 16.

Values of statistical coefficients of disturbancy.

		s	100ρ
Suicides.............	different years	5 000 000	20.33 ± 2.91,
Divorces............	" "	5 000 000	18.81 ± 2.70,
Drownings...........	" "	5 000 000	11.62 ± 1.67,
Deaths	" "	5 000 000	6.99 ± 0.92,
Marriages...........	" "	5 000 000	5.49 ± 0.79,
Births...............	" "	5 000 000	4.07 ± 0.66,
Births of twins.....	" "	5 000 000	4.17 ± 0.68,
Births in Lund......	" "	16 000	3.67 ± 0.59,
Births of boys......	different provinces	12 000	0.95 ± 0.14.

We find confirmed here, as we had already supposed, that the outer disturbances are just as strong upon births of twins as upon single births and approximately as strong upon the number of births in Lund as upon the number of births in the entire kingdom.[2]

Most disturbed is the number of suicides and the number of divorces, least is the number of births of boys (by provinces). However, judging from the mean error, disturbing forces must be considered present even in this last case.

38 After the presence of disturbing influences on the statistical series has been established in this manner, it is the task of the statistician to trace the cause of these disturbances. The theory of correlation, which will be treated in the second part of this resumé, gives the general method for this tracing.

2. Since ρ is not significantly greater for the frequency of marriage than for the frequency of birth, it is suggested that possibly the same disturbing influences, at least partly, affect these two events. This method of proof, however, should not be generalised.

38 However, certain conclusions about the nature of the disturbing influences to which the statistical object is exposed may be obtained solely by observation of the statistical series.

According to the theorem of LEXIS these disturbances consist of changes in the probability of the occurrence of an event from one 'trial' to the next (in this case from one year to the next). These changes, insofar as they are regular, may be treated mathematically as divided into two principal groups: secular and periodic changes.[3] The determination of the latter is beset with considerable difficulty--especially if the length of the period or periods is unknown--, whereas the secular changes of the fundamental probabilities may be determined relatively easily.

39 It is assumed that the fundamental probabilities p_1, p_2, ..., p_N are continuously increased (or decreased) by the same amount β from trial to trial (*ex.gr.* from year to year). The elements of the statistical series are then on the average increased (or decreased) from one trial to the next by the amount $s\beta$, where s is the number of comparison. Theory shews that the number $s\beta$ may then be computed from the formula

$$s\,\beta = \frac{12}{N\,(N^2-1)}\;\Sigma\left(k - \frac{N+1}{2}\right)(m_k - M)$$

40 As example for the application of this formula we shall compute the secular increase of suicides in Sweden from the figures given in Table 14 . The complete treatment of these figures is to be seen in Table 17 .

Because here

$$\frac{N\,(N^2-1)}{12} = 1300,$$

we have

$$s\,\beta = 21\,125 : 1300 = 16.25,$$

so that the number of suicides in Sweden has on the average increased by 16 cases per 5 000 000 inhabitants per year.[4]

41 After the secular change in the elements has been determined, its influence may be eliminated from the statistical series and a new series, free from secular influence, may be derived. From this the periodic (or more generally the oscillatory = not uniform)

3. Called by LEXIS <15 , p. 33> evolutory (more generally symptomatic) and periodic (more generally oscillatory) changes.

4. I have proved that, when the outside influences have essentially secular character, β may be approximately computed, by a simple formula, from the coefficient of disturbancy ρ.

Table 17.

Computation of the secular increase of the number of suicides in Sweden.

$s = 5\,000\,000$, $N = 25$, $M = 600$.

Year	k	m_k	$m_k - M$	$k - \frac{N+1}{2}$	$\left(k - \frac{N+1}{2}\right)(m_k - M)$
1876	1	462	− 138	− 12	+ 1 656
1877	2	479	− 121	− 11	+ 1 331
1878	3	453	− 147	− 10	+ 1 470
1879	4	478	− 122	− 9	+ 1 098
1880	5	421	− 179	− 8	+ 1 432
1881	6	420	− 180	− 7	+ 1 260
1882	7	526	− 74	− 6	+ 444
1883	8	510	− 90	− 5	+ 450
1884	9	464	− 136	− 4	+ 544
1885	10	494	− 106	− 3	+ 318
1886	11	601	+ 1	− 2	− 2
1887	12	541	− 59	− 1	+ 59
1888	13	595	− 5	0	0
1889	14	561	− 39	+ 1	− 39
1890	15	637	+ 37	+ 2	+ 74
1891	16	638	+ 38	+ 3	+ 114
1892	17	708	+ 108	+ 4	+ 432
1893	18	698	+ 98	+ 5	+ 490
1894	19	791	+ 191	+ 6	+ 1 146
1895	20	754	+ 154	+ 7	+ 1 078
1896	21	739	+ 139	+ 8	+ 1 112
1897	22	760	+ 160	+ 9	+ 1 440
1898	23	718	+ 118	+ 10	+ 1 180
1899	24	774	+ 174	+ 11	+ 1 914
1900	25	777	+ 177	+ 12	+ 2 124
					+21 125

changes of the fundamental probabilities may be more easily studied and determined than from the original statistical series.

If we have succeeded in determining the mathematical character of the secular and periodic influences, we have taken an important step toward investigating the foreign factors which disturb the statistical object studied. The closer determination of these factors, as we have already said, is accomplished by means of the theory of correlation.

Cap. VIII. The reduced statistical Series.

42 Let

$$(1) \qquad m_1,\ m_2,\ m_3,\ \ldots,\ m_N$$

be a given statistical series with the corresponding numbers of comparison

$$s_1,\ s_2,\ s_3,\ \ldots,\ s_N.$$

Multiplying the elements in (1) by s/s_1, s/s_2, s/s_3, ..., s/s_N, we obtain a new series

$$(2) \qquad \frac{s}{s_1}m_1,\ \frac{s}{s_2}m_2,\ \frac{s}{s_3}m_3,\ \ldots,\ \frac{s}{s_N}m_N,$$

which is called the reduced statistical series, or the series reduced to the basis s.

We shall compare (2) with the series that we would obtain if s had been the number of comparison for all the elements. Let

$$(3) \qquad m_1',\ m_2',\ m_3',\ \ldots,\ m_N'$$

be this series.

We designate the mean and dispersion of (2) by M and σ, those of (3) by M' and σ'.

Under the hypothesis that the object observed follows the laws of BERNOULLI, it can be proved that

$$(4) \qquad \begin{aligned} M &= M', \\ \sigma &= f_1\,\sigma' = f_1\sqrt{s\,p\,q}, \end{aligned}$$

where

$$(4^*) \qquad f_1 = \sqrt{\frac{1}{N}\left(\frac{s}{s_1}+\frac{s}{s_2}+\frac{s}{s_3}+\cdots+\frac{s}{s_N}\right)}.$$

Although, then, the means of the two series (2) and (3) will agree on the average, the dispersions in general will differ.

Formula (4) gives us the BERNOULLI dispersion of the reduced statistical series (2). Therefore we must multiply the values of σ_B that we obtained in the previous chapter, where the number of comparison varied from trial to trial, by the factor f_1, in order to give the correct value of the BERNOULLI dispersion.

The number of comparison s, to which the numbers from the official statistics were reduced, was however so chosen there that the factor of reduction f_1 differs only insignificantly from unity.

We find from (4*) that f_1 has the value 1 when s is so chosen that

$$\frac{1}{s} = \frac{1}{N}\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \cdots + \frac{1}{s_N}\right),$$

i.e. when s is equal to the harmonic mean of the numbers of comparison $s_1, s_2, s_3, \ldots, s_N$. It is known that the harmonic mean is always smaller than the arithmetic mean of the same quantities, but that its deviation from the arithmetic mean is in general insignificant. In practice, then, we may say that the factor of reduction f_1 is near 1, if s has a value which is near the arithmetic mean of the numbers of comparison $s_1, s_2, s_3, \ldots, s_N$. In those examples of official statistics which were treated in the previous chapter, s was chosen in this manner.

Taking for example the statistical series of the numbers of deaths, drownings, etc. for the years 1876-1900 in Art. 39, the numbers given in the official statistics of Sweden were reduced to a population of 5 000 000 (=s), whereas the actual population (s_k) varied between 4 430 000 and 5 136 000. We find from Table 18 that

$$\Sigma \frac{s}{s_k} = 26.318,$$

and therefore

$$f_1 = \sqrt{26.318 : 25} = \sqrt{1.0527} = 1.0260,$$

$\sqrt{spq}$ is to be multiplied by this number to obtain the Bernoulli dispersion. Correspondingly, the Lexis ratios in Table 15 are to be lessened by division by 1.026. The supernormal dispersion is not notably lessened thereby.

It is not altogether necessary, however, to reduce the given statistical series of elements with varying numbers of comparison in such wise that f_1 is near unity, although this reduction often has certain advantages. The number of

Table 18.

$s = 5\,000\,000.$

43

Year	s_k	$s : s_k$
1876	4 430 000	1.129
1877	4 485 000	1.115
1878	4 532 000	1.108
1879	4 579 000	1.092
1880	4 566 000	1.095
1881	4 572 000	1.094
1882	4 579 000	1.092
1883	4 604 000	1.086
1884	4 644 000	1.077
1885	4 683 000	1.068
1886	4 717 000	1.060
1887	4 738 000	1.055
1888	4 748 000	1.053
1889	4 774 000	1.047
1890	4 785 000	1.045
1891	4 803 000	1.041
1892	4 807 000	1.040
1893	4 824 000	1.036
1894	4 873 000	1.026
1895	4 919 000	1.016
44 1896	4 963 000	1.007
1897	5 010 000	0.998
1898	5 063 000	0.988
1899	5 097 000	0.981
1900	5 136 000	0.974
		26.318

44 comparison (s) of the reduced series may be chosen *ad lib.* Computing the dispersion of the reduced series in the customary manner and comparing it with the BERNOULLI dispersion given by the formula

(5) $$\sigma_B = f_1 \sqrt{s p q}$$

we obtain the LEXIS ratio (L) and the coefficient of disturbancy (ρ) from the formulae of the previous chapter. As an example I shall compute the number of births of twins per 1000 single births from the official statistics of Sweden for the year 1883 .

The elements of the reduced series are found in the fifth column of Table 19 . They give the number of births of twins per 1000 single births in the 25 provinces of Sweden (Gotland is not here excluded). The characteristics (M and σ) of the reduced series are

$$M = M_0 + b = 14 + 16.2 : 25 = 14.65,$$

$$\sigma = \sqrt{85.3 : 25 - b^2} = \sqrt{2.990} = 1.73.$$

Further, the factor of reduction f_1 has the value

$$f_1 = \sqrt{5.758 : 25} = \sqrt{0.2303} = 0.480.$$

In order to compute the BERNOULLI dispersion, we must in addition know the probability (p) of birth of twins, which, according to Table 19 , is

$$p = M : s = 14.65 : 1000 = 0.01465$$

From formula (5) we now have

$$\sigma_B = 0.480 \sqrt{1000 \times 0.01465 \times 0.98535} = 1.824.$$

The LEXIS ratio (L) has therefore the value

$$L = 1.73 : 1.82 = 0.95 \pm 0.14,$$

where the mean error of L is also given. The dispersion is subnormal, but not definitely so. We shall find below a better method to compute it.

45 The reduced statistical series (2) has--as representing a directly observed series with number of comparison s--a conspicuous weakness. Assume that one time s_1 = 10 000 drawings, and that another time s_2 = 100 drawings are made from a pack of cards, and m_1

Table 19.

Reduced series.

Births of twins per 1000 single births in different provinces.

s_k = Number of single births
m_k = number of births of twins
$m'_k = \frac{s}{s_k} m_k$

$s = 1000,\ N = 25,\ M_0 = 14.$

k	s_k	m_k	$s : s_k$	m'_k	$m'_k - M_0$	$(m'_k - M_0)^2$
1	6 251	82	0.160	13.1	− 0.9	0.8
2	4 280	62	0.234	14.5	+ 0.5	0.2
3	3 328	48	0.300	14.4	+ 0.4	0.2
4	4 205	64	0.238	15.2	+ 1.2	1.4
5	7 038	98	0.142	13.9	− 0.1	0.0
6	5 650	77	0.177	13.6	− 0.4	0.2
7	4 895	71	0.204	14.5	+ 0.5	0.2
8	6 276	84	0.159	13.4	− 0.6	0.4
9	1 060	21	0.943	19.8	+ 5.8	33.6
10	4 340	62	0.230	14.3	+ 0.3	0.1
11	6 291	98	0.159	15.6	+ 1.6	2.6
12	10 023	168	0.100	16.8	+ 2.8	7.8
13	3 668	65	0.272	17.7	+ 3.7	13.7
14	7 886	106	0.127	13.5	− 0.5	0.2
15	7 201	114	0.139	15.8	+ 1.8	3.2
16	6 824	98	0.147	14.4	+ 0.4	0.2
17	6 563	94	0.152	14.3	+ 0.3	0.1
18	5 039	75	0.198	14.9	+ 0.9	0.8
19	3 788	56	0.264	14.8	+ 0.8	0.6
20	5 639	87	0.177	15.4	+ 1.4	2.0
21	6 131	82	0.163	13.4	− 0.6	0.4
22	6 308	87	0.159	13.8	− 0.2	0.0
23	2 883	29	0.347	10.1	− 3.9	15.2
24	3 748	57	0.267	15.2	+ 1.2	1.4
25	3 334	46	0.300	13.8	− 0.2	0.0
			5.758		+16.2	+85.8

and m_2 black cards obtained. If we wish to compute from this the number of black cards in $s = 1000$ drawings, we obtain the values

$$\frac{s}{s_1} m_1 \quad \text{and} \quad \frac{s}{s_2} m_2.$$

45 But these values are obviously not equally valid. The mean error of the latter value is indeed ten times as large as that of the former. In series (2), however, each element, whatever its number of comparison, influences the computation of M and σ to the same extent.

46 One is therefore led to consider a series which is so compounded that the element sm_k/s_k occurs s_k times ($k = 1, 2, \ldots, N$). In this manner is obtained a series with $s_1 + s_2 + s_3 + \ldots + s_N$ elements, which is called the reduced and weighted statistical series. To distinguish it, the series (2) may be called the simple reduced series. If M_1 and σ_1 are the mean and dispersion of the latter, M_2 and σ_2 those of the former, then the following theorems hold:

1. On the average $M_1 = M_2$.
2. On the average the mean error of M_1 is larger than that of M_2.

For the numerical computation of M_2 and σ_2 we have the formulae

$$M_2 = \frac{s\,\Sigma m_k}{\Sigma s_k}, \tag{7}$$

$$\sigma_2 = f_2 \sqrt{\frac{1}{N}\Sigma \frac{s}{s_k}(m_k - p_0 s_k)^2}, \tag{8}$$

where p_0 is given by (11) and

$$f_2 = \sqrt{\frac{s\,N}{\Sigma s_k}}. \tag{9}$$

It can be proved that f_1 (in formula (4*)) is always smaller than f_2.

It is most convenient, however, to compute σ_2 from the average deviation θ, given by the formula

$$\theta = f_2^2 \frac{1}{N} \Sigma \,|\, m_k - p_0 s_k \,| \tag{10}$$

where

$$p_0 = \Sigma m_k : \Sigma s_k = M : s \tag{11}$$

and where, as usual, $|m_k - p_0 s_k|$ signifies that all differences between m_k and $p_0 s_k$ are to be taken positive. Then

$$\sigma_2 = 1.2533\,\theta \tag{11*}$$

It is noteworthy, as is seen from formulae (7) and (10), that both the mean and the dispersion of the reduced and weighted series may be had directly from the observed series

$$m_1, m_2, m_3, \ldots, m_N$$

without first constructing the reduced series. The computation of the characteristics of the reduced and weighted series is therefore very simple.

Table 20.

Reduced and weighted series.
Births of twins per 1000 single births in different provinces.

s_k = number of single births
m_k = number of births of twins

$s = 1000,\ N = 25.$

k	s_k	m_k	$p_0\, s_k$	$\lvert m_k - p_0\, s_k \rvert$
1	6 251	82	91.0	9.0
2	4 280	62	62.3	0.3
3	3 328	48	48.5	0.5
4	4 205	64	61.2	2.8
5	7 038	98	102.5	4.5
6	5 650	77	82.3	5.3
7	4 895	71	71.3	0.3
8	6 276	84	91.4	7.4
9	1 060	21	15.4	5.6
10	4 340	62	63.2	1.2
11	6 291	98	91.6	6.4
12	10 023	168	145.9	22.1
13	3 668	65	53.4	11.6
14	7 886	106	114.8	8.8
15	7 201	114	104.8	9.2
16	6 824	98	99.4	1.4
17	6 563	94	95.6	1.6
18	5 039	75	73.4	1.6
19	3 788	56	55.2	0.8
20	5 639	87	82.1	4.9
21	6 131	82	89.3	7.3
22	6 308	87	91.8	4.8
23	2 883	29	42.0	13.0
24	3 748	57	54.6	2.4
25	3 334	46	48.5	2.5
	132 649	1931		135.3

47 As example for the application of these formulae I take the series that was treated in Art. 44 as a simple reduced series.

By formula (11) we find from the second and third columns of Table 20

47 $$p_0 = 1931 : 132\,649 = 0.01456$$

as relative probability of birth of twins. Multiplying p_0 by the numbers s_k we obtain the numbers of the fourth column. We further obtain

$$M_2 = 1000\,p_0 = 14.56$$

and

$$f_2 = \sqrt{\frac{25 \times 1000}{132\,649}} = 0.4341$$

and by (10)

$$\theta = (0.4341)^2 \times 135.3 : 25$$
$$= 1.020,$$

so that by (11*)

$$\sigma_2 = 1.278$$[1].

Comparison with the results reached for the same series as a simple reduced series in Art. 44 shews that the means correspond well and that $\sigma_2 < \sigma_1$.

48 The BERNOULLI dispersion for the reduced and weighted series is obtained from the formula

$$\sigma_B = f_2 \sqrt{s\,p_0\,q_0}$$

so that in this example

$$\sigma_B = 0.4341 \sqrt{1000 \times 0.01456 \times 0.98544} = 1.644,$$

so that the LEXIS ratio has the value

$$L = 1.278 : 1.644 = 0.777$$

as I already stated in Art. 35 .

49 For the mean error of M_2 we obtain the value

$$\varepsilon(M_2) = \frac{\sigma_2}{\sqrt{N}} = \frac{1.278}{\sqrt{25}} = 0.256,$$

whereas the arithmetic mean M_1 of the simple reduced series (computed in Art. 44) has the mean error

1. Notice that σ_2 signifies the observed dispersion.

$$\varepsilon(M_1) = \frac{\sigma_1}{\sqrt{N}} = \frac{1.73}{\sqrt{25}} = 0.346$$

so that, in correspondence with theory, M_2 is more reliable than M_1.

50 As one realises from the example treated above, the reduced and weighted series is convenient to use in numerical computation. The sums Σs_k and Σm_k are, and were already in the days of SÜSSMILCH, customarily given in statistical tables, and a good part of the computation is already done thereby.

Neverthaless the simple reduced series may be preferred when

1. The factor of reduction f_1 (or f_2) is near unity, and when
2. the number of elements is so large that a division into classes must be undertaken.

It is further to be remarked that the theoretical excellence of the reduced and weighted series is only proven for series with normal (or nearly normal) dispersion. For series with supernormal dispersion--and these form the largest part of the series in applied statistics--more exact characteristics can be derived, although even for these the reduced and weighted series serves well. However, whether the simple reduced or the reduced and weighted series is used, it is well not to proceed from elements whose numbers of comparison (s_k) differ too widely.

Part Two.

Heterograde or qualitative Statistics.

Cap. IX. Introduction.

51 In the previous chapters I have treated the two first characteristics--the mean and the dispersion--of a homograde statistical series, and shewn how their numerical values may be derived from the elements of the series. Also I have displayed, by means of examples from official statistics, the significance of the theorems of BERNOULLI, POISSON, and LEXIS for the interpretation of the values found for the characteristics.

The essence of our investigations was to obtain a measure of the disturbing outside influences upon the statistical object, influences which in general cause the fluctuations of the numerical value of the elements of the observed statistical series to fail to obey those simple laws abstracted from cases of definite drawings of cards from a pack and from such other experiments as lead to the Bernoullian laws of probability.

To continue our treatment of homograde statistics it were necessary

1. to investigate the higher characteristics of statistical series,
2. to treat the connexion between simultaneously presented homograde statistical series, *i.e.* the theory of correlation for homograde quantities.

In order to save space, I shall treat these two questions, as well as the related problem of frequency curves, together for the two parts of statistics. This juxtaposition is practicable after the theory of LEXIS and the properties of the reduced series have been treated, since they find application only in homograde statistics. I shall however present in the sequel several points at which the homograde statistical series requires special treatment.

52 Before going on let us observe an example of a heterograde statistical series. I take the figures for rainfall in Lund for the years 1899-1908 .

The numbers in the second column of the table give the rainfall in millimeters for each of the years 1889-1908 . Each of

Table 21.

Rainfall in Lund.

$N = 20, M_0 = 600.$

x = total annual rainfall in millimeters.

Year	x	$x - M_0$		$(x - M_0)^2$
1889	567		− 33	1 100
1890	594		− 6	0
1891	720	+ 120		14 400
1892	615	+ 15		200
1893	625	+ 25		600
1894	724	+ 124		15 400
1895	673	+ 73		5 300
1896	703	+ 103		10 600
1897	648	+ 48		2 300
1898	728	+ 128		16 400
1899	511		− 89	7 900
1900	661	+ 61		3 700
1901	597		− 3	0
1902	541		− 59	3 500
1903	663	+ 63		4 000
1904	563		− 37	1 400
1905	607	+ 7		0
1906	576		− 24	600
1907	530		− 70	4 900
1908	717	+ 117		13 700
		+ 884	− 321	106 000

these numbers forms an element of the statistical series. The total number N of elements is 20 .[1]

The mean and dispersion of the series are computed in exactly the same manner as for a homograde series. We choose a provisional mean, here taken at 600 , write the numbers $x-M_0$ and $(x-M_0)^2$ and take the sums. In view of the small number of elements it suffices to round the squares $(x-M_0)^2$ to hundreds. We then have, by the rules of the previous chapter,

1. Notice that a datum that must not be missing when homograde statistical series are treated is missing here; the number of comparison (s). Indeed, when heterograde elements are discussed, a 'number of comparison' can never be given.

52

$$b = (884 - 321) : 20 = +28.2,$$

$$M = 600 + b = 628.2,$$

$$\sigma = \sqrt{106\,000 : 20 - (28.2)^2} = 67.1.$$

Computing the mean errors by the formulae of Cap. IV, we have

$$M = 628.2 \pm 15.0,$$

$$\sigma = 67.1 \pm 10.6.$$

If the number N of elements were large, a division into classes would be advised. The process to be applied here, likewise the computation of the characteristics, is exactly the same as we have given in Cap. III for a homograde statistical series. The distinction between the two kinds of series lies elsewhere.

53 Whereas in homograde statistics the grouping of the elements about the mean can be completely satisfactorily explained by the theorems of BERNOULLI, POISSON, and LEXIS, the situation with heterograde statistical series is completely different. No mathematical argument, for example, suffices to predict the mean deviation of temperatures in Lund for the month of May, and no *a priori* conclusions suffice to estimate, even approximately, the dispersion of a statistical series concerned with the height of adults (of a certain race). Not only is it not possible to make such *a priori* estimates of the dispersion (or of other characteristics) of a heterograde series, but it even appears doubtful whether in general the elements of a series of heterograde individuals are grouped according to a normal law that can be described by simple mathematical formulae.

Experience has however shewn that at least this last is the case. It has indeed been found that a statistical series of heterograde individuals possesses essentially the same properties as a similar series of homograde individuals. The principal distinction is that whereas the characteristics of the latter series can be computed--to a better or worse approximation--from some simple theorems of the mathematical theory of probability, these theorems desert us completely for a series of heterograde individuals.

We are here compelled to seek other means of explanation. The first step is to seek a sufficiently general hypothesis to explain how the deviation that an individual of the population shews with respect to the attribute to be investigated may be composed. The hypothesis upon which I have built my investigations is that each individual deviation may be considered as the sum of a set of un-

known but small quantities, which are called elementary errors.[2]

54 Observe, for example, the height in a population of grown men. If it can be assumed that all men of the population have completely similar ancestry; that they have been exposed to exactly the same upbringing, food, climatic influences; that all other circumstances, which might affect the height of man, have been identical for all men of the population: then we must conclude, as surely as an effect is determined by its cause, that the height of all these men would be the same. The differences in the heredity[3], upbringing, food, etc., may here be considered as so many sources of error with respect to the height of these men. Each source of error causes a positive or negative elementary error in the height. The resulting deviation of the height of an individual from the ideal height that they would all have, if they had been exposed to the same influences, consists of the sum of all these small quantities. Obviously the number of sources of error must be considered as very large or infinite.

55 To derive from the hypothesis of elementary errors the laws that hold for the distribution of the elements in a heterograde statistical series is an assignment in the field of mathematical probability theory. To give without mathematical arguments a presentation of the train of thought that can be followed in this connexion is by no means easy; I do not intend here to make a such attempt. I confine myself to pointing out that LAPLACE has first shewn how such problems may be solved, although he did not carry the analysis as far as is required for the derivation of the general laws.[4]

56 Some preliminary remarks are in order about the terminology here used. We assume that the elements are arranged in classes. Let *x* be the class mark and *y* the number of elements belonging to the class with the mark *x*. It is then possible to derive from the mathematical theory a formula, or mathematical expression, of such form in *x* and *y*, that *y* can be computed analytically for any value of *x*. This expression is called a frequency function. If *x* and *y* are graphically presented, plotting *x* as the abscissa and *y* as the ordinate, the result is known as a frequency curve.

It can be shewn that these frequency curves (frequency functions) occur in two different forms, which I call frequency curves (frequency functions) of Type A and of Type B.

2. The hypothesis as such is not new, but was used as early as 1837 by HAGEN, in a somewhat more special form, to derive the Gaussian law of error. <11 , p. 29>

3. 'Heredity' of course here includes an infinite class of sources of error; any attempt to classify these more closely is here superfluous.

4. See <2>, <3>.

56 I shall explain some properties of these types in Capp. XI. and XII. If the higher characteristics are small, the frequency curves of Type A approach the Gaussian curve of error, which will be studied in the following chapter under the name normal curve.

Cap. X. The normal frequency Curve.

57 We found in the third chapter that the dispersion σ can serve to illuminate the manner in which the elements of a statistical series are grouped about the mean M. We made hereon the observations that the number of elements lying between the limits $M+\sigma$ and $M-\sigma$ is about two thirds of the total number, and further that all the elements generally lie between the bounds $M - 3\sigma$ and $M + 3\sigma$. Therefore the elements obviously lie more densely in the neighbourhood of the mean than elsewhere. We can conclude from the second of our observations that the number of elements declines rapidly when one goes away from the mean.

If the higher characteristics are small, as is assumed throughout this chapter, we can predict much more exactly how the elements of the series are distributed about the mean.

We shall always assume that the elements are arranged in classes. Let w be the class interval, x the class mark, and y the number of elements belonging to the class x; then the normal distribution of elements follows the simple law

$$y = \frac{Nw}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2 w^2}{2\sigma^2}} \tag{1}$$

By this formula[1] the theoretic number, y, of elements belonging to the class x, can be calculated.

58 In graphic portrayal of the normal curve it is useful so to choose the units (the scale) of the x- and y-coordinates, that all normal curves are directly comparable. This is accomplished by introducing normal coordinates X and Y, defined by the equations [2]

$$\begin{aligned} X &= \frac{(x-b)w}{\sigma} \\ Y &= \frac{\sigma}{Nw} y \end{aligned} \tag{2}$$

1. Here $bw = M - M_0$, so that b (but not σ) is expressed in class interval units.

2. Thus the unit of X-coordinates is the dispersion.

so that the equation of the normal curve takes the form

$$Y = \varphi_0(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2} \tag{3}$$

The function $\varphi_0(X)$ is called the probability function. Its value is given in the following table to three places of decimals.

Table 22.

The probability function.

X	$\varphi_0(X)$	X	$\varphi_0(X)$	X	$\varphi_0(X)$	X	$\varphi_0(X)$
0.0	0.399	1.0	0.242	2.0	0.054	3.0	0.004
0.1	0.397	1.1	0.218	2.1	0.044	3.1	0.003
0.2	0.391	1.2	0.194	2.2	0.036	3.2	0.002
0.3	0.381	1.3	0.171	2.3	0.028	3.3	0.002
0.4	0.368	1.4	0.150	2.4	0.022	3.4	0.001
0.5	0.352	1.5	0.130	2.5	0.018	3.5	0.001
0.6	0.333	1.6	0.111	2.6	0.014	3.6	0.001
0.7	0.312	1.7	0.094	2.7	0.010		
0.8	0.290	1.8	0.079	2.8	0.008		
0.9	0.266	1.9	0.066	2.9	0.006		

If we wish to compare an observed statistical series with the normal curve, we must--after calculating the characteristics of the series--derive the values of the normal coordinates by means of formulae (2), where the observed number of elements belonging to the class x is to be substituted for y. If, moreover, we use Table 22 to construct the normal curve, the computation of normal coordinates and their imposition on the diagram is the work of a few minutes. I advise my reader to carry out such a construction himself; the work involved is simple and moreover gives a concrete picture of the manner in which the elements of a statistical series are distributed. Here, as with statistical calculations in general, the arithmetic is best done with a computing machine.

59 As first example I shall treat a homograde statistical series. I choose for this purpose the series that has been used several times already in illustration of statistical theorems; the number of boys per 500 births in Sweden (for different months and provinces). The series is arranged in classes in Table 5 ; data derived therefrom are given in the first two columns of Table 23 . We find the computation of b and σ elsewhere. The normal

59 coordinates for each class are now computed by means of formulae (2), and the corresponding points shewn on the diagram by small circles.

Table 23.

Number of boys per 500 births.

$N = 576$, $w = 5$.

$b = +0.024\,w$, $\sigma = 2.498\,w$

$= +0.12$ $= 12.49$.

	x	F (x)	X	Y
	−11	1	−4.41	.004
	−10	0	−4.01	.000
60	− 9	0	−3.61	.000
	− 8	1	−3.21	.004
	− 7	2	−2.81	.009
	− 6	5	−2.41	.022
	− 5	13	−2.01	.056
	− 4	18	−1.61	.078
	− 3	47	−1.21	.205
	− 2	60	−0.81	.260
	− 1	81	−0.41	.351
	0	108	−0.01	.468
	+ 1	91	+0.39	.395
	+ 2	60	+0.79	.260
	+ 3	44	+1.19	.191
	+ 4	22	+1.59	.095
	+ 5	16	+1.99	.069
	+ 6	6	+2.39	.026
	+ 7	0	+2.79	.000
	+ 8	0	+3.19	.000
	+ 9	1	+3.59	.004

As Fig. 1 shews, the observations follow the normal curve, given by the solid line, very closely. The largest discrepancy occurs at $x = 0$, so that the number of boys in 500 births is found somewhat more often in the neighbourhood of the mean (257) than could be expected from the normal frequency curve. Observe further that the departures from the normal curve shew no definite systematic course. The departures may in general be considered accidental. The number of observations is not large enough to determine with certainty whether the frequency curve for this statistical object depart from the normal form.

As example of a heterograde statistical series I chose the periods of gestation in cows, according to observations in the agricultural institute at Alnarp, which have very kindly been placed at my disposal. The period is counted from the date of covering to the birth of the calf. The mean value of the period was 278.15 days and the dispersion was 5.35 days.[4] The number of cases registered was 393 .

We find (from Table 24 and Fig. 2) that the number of elements in the different classes is again arranged in accordance with the normal curve. The deviations are somewhat greater than in the previous example, doubtless because of the small number of elements. They have, however, an accidental character, and we are not in a position to decide from the material at hand whether, or in what way, the frequency curve for the period of gestation of cows deviate from the normal form.

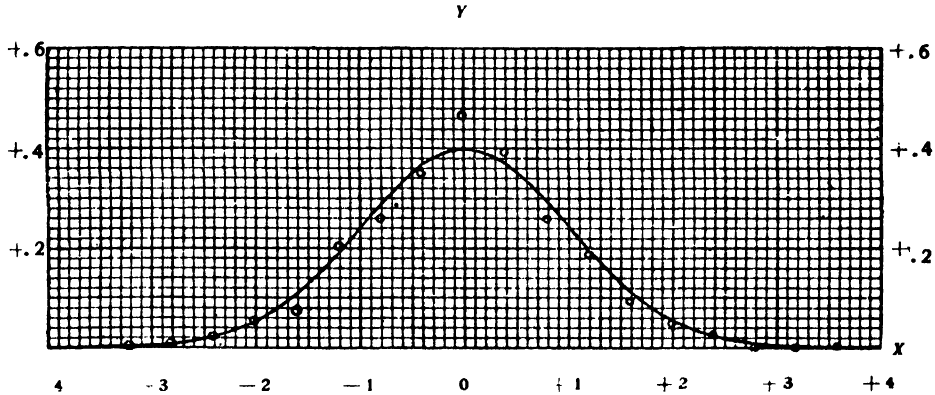
61 The more elements in the statistical series, the more uniform the distribution of deviations from the normal form. In most

4. With mean error added, $M = 278.15 \pm 0.27$, and $\sigma = 5.85 \pm 0.19$.

cases, again, the correspondence with the normal curve becomes better the more numerous the elements are. This, of course, is

Number of boys per 500 births.

Fig. 1.



particularly true where there are grounds to assume that the phaenomenon in question follows the simple probability laws of BERNOULLI. Taking, for example, the series in Table 10 with 1000 elements (number of black cards in 10 drawings), we get the diagram Fig. 3 .

The agreement with the normal curve is here practically complete. If, instead of this, one were to take the series of Table 9 (number of black cards in 50 drawings), which contains only 200 elements, one would have to expect significantly greater deviations from the normal curve. Obviously these deviations must here be considered totally accidental.

Table 24.

Period of gestation in cows.

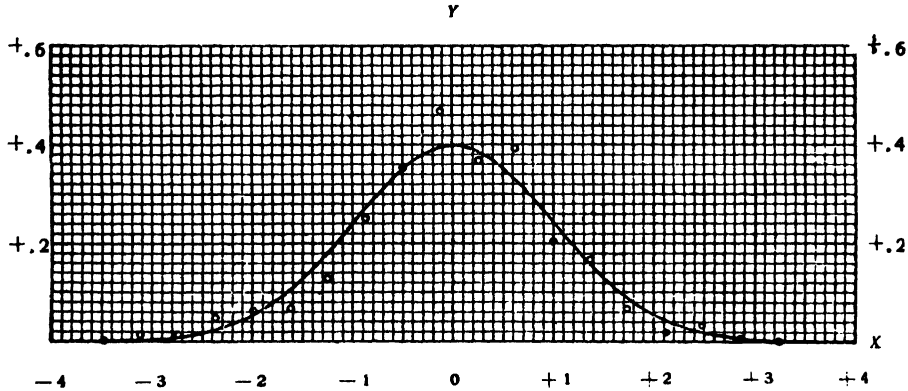
$N = 393$, $w = 2$ days,
$M_0 = 277.5$, $\sigma = 2.676\, w$,
$b = +0.388$.

x	$F(x)$	X	Y
— 8	2	— 3.11	.014
— 7	2	— 2.74	.014
— 6	7	— 2.37	.048
— 5	9	— 1.99	.061
— 4	10	— 1.62	.068
— 3	19	— 1.25	.129
— 2	37	— 0.87	.252
— 1	52	— 0.50	.354
0	69	— 0.12	.470
+ 1	54	+ 0.25	.368
+ 2	58	+ 0.62	.395
+ 3	30	+ 1.00	.204
+ 4	25	+ 1.37	.170
+ 5	10	+ 1.74	.068
+ 6	3	+ 2.12	.020
+ 7	5	+ 2.49	.034
+ 8	1	+ 2.87	.007

61

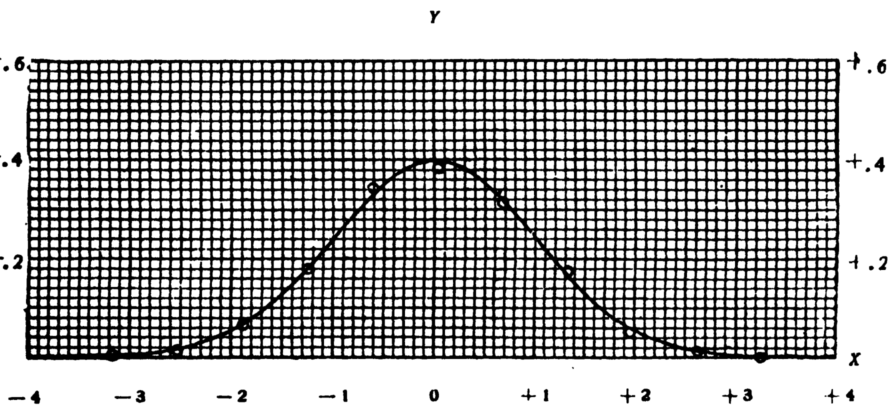
Periods of gestation in cows.

Fig. 2.



Number of black cards in 10 drawings.

Fig. 3.



Cap. XI. Frequency Curves of Type A.

Table 25.

X	ϕ_0	ϕ_3	ϕ_4
-3.5	+0.0009	+0.0283	+0.0694
-3.0	+0.0044	+0.0798	+0.1330
-2.5	+0.0175	+0.1424	+0.0800
-2.0	+0.0540	+0.1080	-0.2700
-1.5	+0.1295	-0.1457	-0.7043
-1.0	+0.2420	-0.4839	-0.4839
-0.5	+0.3521	-0.4841	+0.5501
0	+0.3989	0.0000	+1.1968
+0.5	+0.3521	+0.4841	+0.5501
+1.0	+0.2420	+0.4839	-0.4839
+1.5	+0.1295	+0.1457	-0.7043
+2.0	+0.0540	-0.1080	-0.2700
+2.5	+0.0175	-0.1424	+0.0800
+3.0	+0.0044	-0.0798	+0.1330
+3.5	+0.0009	-0.0283	+0.0694

62 In the preceding chapter we have attempted to shew by some examples how statistical series of heterograde or homograde elements approach more or less a certain form of frequency distribution; which form, graphically portrayed, is known as the normal curve. We have further seen that departures from this form occur, which have in many cases a purely accidental character and vary from one series to another. In other cases, particularly in statistical series that contain a large number of elements (say over 1000), however, these departures display a regular and systematic character, pointing to frequency curves which have a form differing from the 'normal'. I shall give in this chapter some properties of these curves.

It is again assumed that the elements are arranged in classes. It is further assumed that the normal coordinates X and Y are introduced by formulae (2) of the preceding chapter in place of the class mark x and the frequency $F(x)$.

The general equation for frequency curves of Type A is then

$$(1) \qquad Y = \varphi_0 + \beta_3\varphi_3 + \beta_4\varphi_4 + \beta_5\varphi_5 + \cdots,$$

1. <The β_3, β_4, etc., here are not the same as the frequently used β-statistics introduced by PEARSON. --Tr.>

62 where φ_0, or, in full, $\varphi_0(x)$, designates as before the probability function and φ_3, φ_4, etc., represent the derivatives of this function.

I give here an abridged table of these functions, which will in many cases suffice for the graphic construction of a frequency curve.

Multiplying the numbers of the third column by β_3 and those of the fourth column by β_4 and adding these products to the numbers of the second column, we obtain the theoretic Y-coordinates. Joining by a line the points obtained in this wise, we have the theoretic frequency curve. The observed frequency curve is obtained by computing the normal coordinates for each class, according to formulae (2) of the preceding chapter.

The diagram is adapted from that used at the Lund observatory. The printed curve is the normal curve. In the computations under the diagram the first four lines give the observed normal coordinates, computed for each class x by formulae (2) of the preceding chapter; the five succeeding lines give the theoretic Y-coordinates, which for simplicity's sake are computed for X = -3.5 , -3.0 , -2.5 , etc.

If the characteristics β_3 and β_4 are not known, the normal curve and the first four lines under the diagram suffice.

63 The coefficients β_3, β_4, β_5, etc., may be considered together with M and σ as the characteristics of the given frequency curve. For practical reasons, however, it is advantageous somewhat to modify the definition of these characteristics (*cf.* the next section).

The calculation of the coefficients β_3, β_4, β_5, etc., from the given statistical series can be performed by elementary methods. I give on an adjacent page the complete schema with appropriate checks, as given in [4]. The whole computation takes, for a not particularly practised computer, about an hour.

64 When the number of elements in the statistical series is not exceptionally large,[2] it suffices in general to compute the coefficients β_3 and β_4. The characteristics which these two coefficients determine are called the skewness S and the excess E, and are defined by the relations

$$(2)\qquad \begin{aligned} S &= 3\,\beta_3, \\ E &= 3\,\beta_4 \end{aligned}$$

The skewness S--also called the coefficient of asymmetry--gives, as the name suggests, a skew form to the frequency curve, so that the elements no longer, as with the normal curve, are

2. <EDGEWORTH <7> recommends the addition of the term $\frac{1}{2}\beta_3^{\,2}\phi_6$. When this term is used, the excess E is to be replaced by $E_E = 3\beta_4 - 7\frac{1}{2}\beta_3^{\,2}$. --Tr.>

Statistical object:

Schema for the Computation of Characteristics.

Check.

$(x+1)^4$		x	$F(x)$	$x\,F(x)$	$x^2\,F(x)$	$x^3\,F(x)$	$x^4\,F(x)$	$(x+1)^4(Fx)$	
6 561		-10							
4 096		-9							
2 401		-8							
1 296		-7							
625		-6							
256		-5							μ_4'
81		-4							$4\,\mu_3'$
16		-3							$6\,\mu_2'$
1		-2							$4\,\mu_1'$
0		-1							μ_0'
		Σ_1							Σ_3
1		0							
16		$+1$							
81		$+2$							
256		$+3$							
625		$+4$							
1 296		$+5$							
2 401		$+6$							
4 096		$+7$							
6 561		$+8$							
10 000		$+9$							
14 641		$+10$							
		Σ_2							
		μ_0'							$= \Sigma_3$
		ν_0'							
				b	ν_2'	ν_3'	ν_4'		

		b	$b\,\nu_2'$	$b\,\nu_3'$	$b\,\nu_3$
		b^2	$b^2\,\nu_2'$		$b^2\,\nu_2$
		b^3			
		b^4			

		ν_2'	ν_3'	ν_4'	ν_4
		$-b^2$	$-3\,b\,\nu_2'$	$-4\,b\,\nu_3'$	$4\,b\,\nu_3$
		$\nu_2 = \sigma^2$	$2\,b^3$	$6\,b^2\,\nu_2'$	$6\,b^2\,\nu_2$
		σ	ν_3	$-3\,b^4$	b^4
		σ^3	$\nu_3 : \sigma^3$	ν^4	$\Sigma_4 \qquad = \nu_4'$
		σ^4	β_3	$\frac{\nu_4}{\sigma^4} - 3$	
			S	β_4	
				E	

$$\beta_3 = -\frac{\nu_3}{6\,\sigma^3}; \quad \beta_4 = \frac{1}{24}\left(\frac{\nu_4}{\sigma^4} - 3\right);$$

$$S = 3\,\beta_3; \qquad E = 3\,\beta_4.$$

64 symmetrically distributed about the mean. When the skewness is positive there is a larger number ($2SN/3$) of elements greater than the mean than there is of elements smaller than the mean, and conversely when the skewness is negative. The highest point of the frequency curve, which corresponds to that value of the statistical quantity at which the elements are most dense (in the case of a heterograde series), no longer coincides with the mean, but is separated from it by the distance $S\sigma$, on the positive or negative side according to the sign of S. The arithmetic mean no longer gives the most probable value of the attribute; this coincides with the value corresponding to the highest point of the frequency curve.

The excess E does not influence the symmetry of the frequency curve, but alters the distribution, determined by the normal curve, of elements into different classes. If the excess is positive, the number of elements in the neighbourhood of the mean is greater than in a normal distribution. The frequency curve is elevated above the normal curve in the centre (therefore, in the neighbourhood of the mean), whence the name excess. The definition of excess is so chosen that this elevation is equal to the product of E by the height of the normal curve.

The skewness and excess together give very different forms to frequency curves, which recall the normal curve more or less, so long as the characteristics S and E are small.[3] I shall give in the next section some examples of such frequency curves.

65 As first example I choose a series of 12 000 measurements of the width of a kind of brown bean (*Phaseolus vulgaris*), which Professor JOHANNSEN[4] in Copenhagen has very kindly placed at my disposal. The measurements were made in his biologic laboratory.

Computing first the mean and dispersion by the methods of Capp. II.

Table 26.

Width of brown beans.

$N = 12\,000$, $w = 0.25$ mm.
$M_0 = 8.825$.

x	$F(x)$	X	Y
−10	3	−3.53	.000
−9	5	−3.12	.001
−8	24	−2.72	.005
−7	103	−2.82	.021
−6	239	−1.91	.049
−5	624	−1.51	.129
−4	1187	−1.11	.246
−3	1650	−0.70	.341
−2	1883	−0.30	.389
−1	1930	+0.10	.399
0	1638	+0.50	.339
+1	1130	+0.91	.234
+2	737	+1.81	.152
+3	427	+1.71	.087
+4	221	+2.12	.046
+5	110	+2.52	.023
+6	57	+2.92	.012
+7	24	+3.88	.005
+8	6	+3.78	.001
+9	2	+4.18	.000

3. <When a frequency curve is drawn from β_3 and β_4 solely, one or both tails will dip below the X-axis, unless β_3 and β_4 lie within the limits given by Appendix Table 44 . --Tr.>

4. I take here the opportunity of warmly recommending the excellent picture of biological statistics that this distinguished investigator has given <13>.

Statistical object:

$N =$ $\quad$ $M =$ $\quad$ $\sigma =$

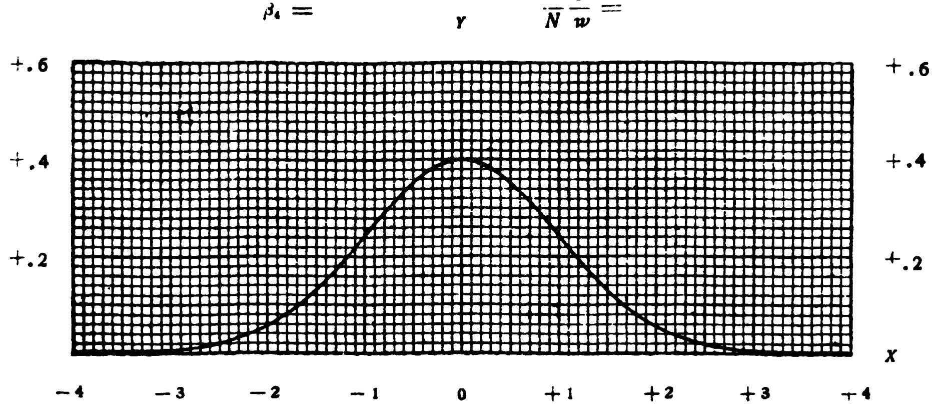
$w =$ $\quad$ $w : \sigma =$ $\quad$ $\sigma : w =$

$\beta_3 =$ $\quad$ $\beta_4 =$

$M_0 =$ $\quad$ $b =$ $\quad$ $\frac{1}{N} =$ $\quad$ $\frac{1}{N}\frac{\sigma}{w} =$

$$X = (x - b)\frac{w}{\sigma},$$

$$Y = \frac{1}{N}\frac{\sigma}{w}F(x).$$



X																								
x	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12
$F(x)$																								
Y																								

X	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	±0	+0.5	+1	+1.5	+2	+2.5	+3	+3.5
φ_0	+.0009	+.0044	+.0175	+.0540	+.1295	+.2420	+.3521	+.3989	+.3521	+.2420	+.1295	+.0540	+.0175	+.0044	+.0009
$\beta_3\varphi_3$															
$\beta_4\varphi_4$															
Y															

 and III., we nave

$$b = -1.253, \quad a = +2.482\,w,$$

so that

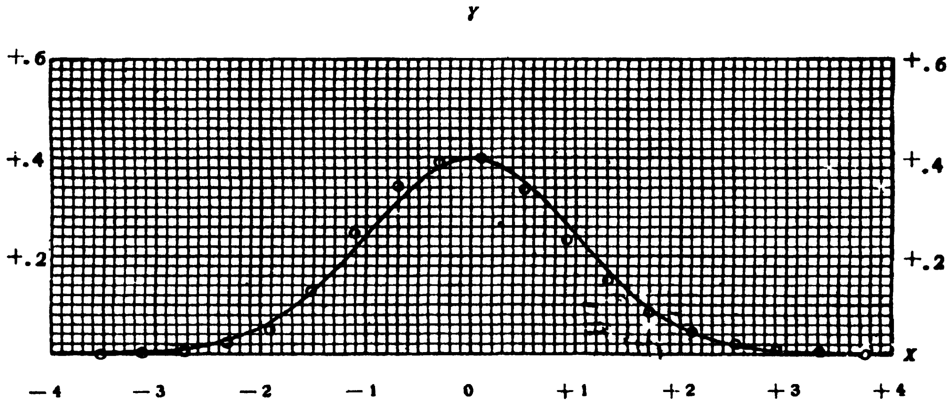
$$M = 8.512 \text{ mm}, \quad \sigma = 0.620 \text{ mm}.$$

By means of the formulae (2) of the preceding chapter the values of the normal coordinates were computed. They are found in Table 26 .

The corresponding points of the observed frequency curve are given in the diagram by small circles.

Width of beans phaseolus vulgaris.

Fig. 4.



One might at first glance be inclined to consider the correspondence with the normal curve as extremely good. A closer observation (which is indeed made somewhat difficult by that the frequency curve has been reproduced here in very small format) shews, however, that the accidental deviations may indeed be considered vanishingly small, but that there obviously exists a systematic deviation of the observations from the normal curve. The observed Y-coordinates (I am here following the curve from left to right) are at first smaller than the Y-coordinates of the normal curve, then from $X = -1.5$ to $X = +0.3$ somewhat larger, then from $X = +0.3$ to $X = +2.0$ smaller again, and finally for higher X-values larger again.

We find that the observed points determine very exactly a curve that differs throughout from the normal. In accord with our earlier definitions of S and E we may say without further computation that the observed frequency curve obviously has a small

negative skewness and a small positive excess.

This is confirmed by the numerical values of S and E. Indeed one finds

$$S = -0.141, \quad E = +0.030.$$ [5]

66 The peak of the frequency curve lies therefore 0.14 in the negative direction from the mean, and the number of elements in the immediate neighbourhood of the mean is as 103 : 100 to that in a normal distribution.

I have on an earlier occasion expounded the need to know the uncertainty of the obtained values of the mean and dispersion. This need is possibly even greater for the skewness and excess.

For frequency curves that differ only insignificantly from the normal, the mean errors of S and E are given by the formulae

$$\varepsilon(S) = \frac{1.2247}{\sqrt{N}}, \qquad \varepsilon(E) = \frac{0.6124}{\sqrt{N}} \tag{3}$$

In the present example, then, where N = 12 000, we have, adding the mean errors,

$$S = -0.141 \pm 0.011,$$
$$E = +0.030 \pm 0.006.$$

We may hence conclude that the values that we have found for the skewness and excess cannot here be considered accidental fluctuations from a normal frequency curve, but that the frequency curve for the width of these brown beans indeed deviates from the normal form in a manner that is characterised by the given values of S and E.

Drawing the theoretic frequency curve from the values of S and E above (applying Table 25), one finds such good agreement with the observed values that the deviations are not noticeable on the scale used in Fig. 4.

67 In *Anthropologia suecica* [21] Professors RETZIUS and FÜRST have studied the Swedish recruits of the years 1897 and 1898 from the point of view of statistical anthropology. From their work I extract the following figures, which relate to the cephalic index for 22 505 Swedish recruits of the year 1897.

5. <E_E = 0.013. --Tr.>

Table 27.

Cephalic index of Swedish recruits.

$N = 22\,505,\ w = 2,\ M_0 = 77.5.$

x	$F(x)$	X	Y
−6	12	−3.42	.001
−5	87	−2.77	.006
−4	510	−2.12	.035
−3	1952	−1.48	.134
−2	4346	−0.83	.298
−1	6039	−0.18	.414
0	5050	+0.47	.346
+1	2822	+1.11	.194
+2	1172	+1.76	.080
+3	377	+2.41	.027
+4	94	+3.06	.006
+5	31	+3.71	.002
+6	13	+4.85	.001

The class interval w is chosen at 2, the provisional mean M_0 taken at 77.5. From the observed frequencies $F(x)$ we have

$$b = -0.721,\quad \sigma = +1.544,$$

expressed in class interval units.

Therefore (returning to the index as unit),

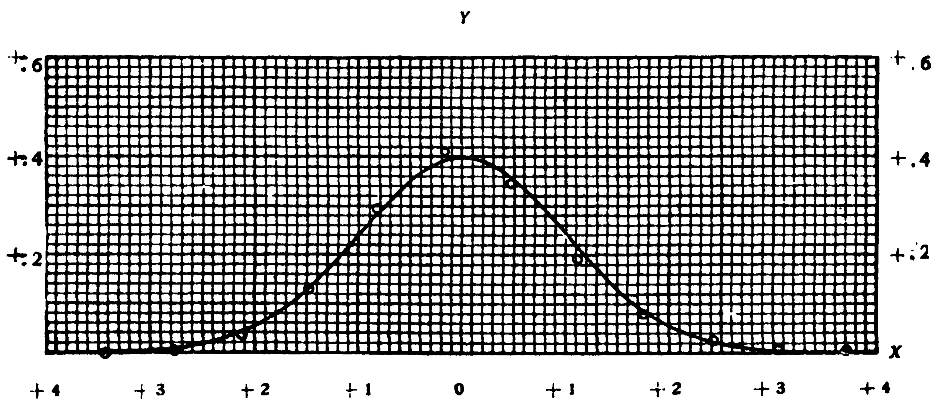
$$M = 77.5 - 1.442 = 76.058 \pm 0.021,$$

$$\sigma = 3.088 \pm 0.015.$$

From b, w, and σ, we compute the normal coordinates given in the third and fourth columns of the adjacent table. The corresponding points are given in Fig. 5 and compared with the normal curve. Considerations analogous

Cephalic index of recruits.

Fig. 5.



to those of the previous example lead us to the conclusion that here again the frequency curve must have a negative skewness and a positive excess. Indeed numerical computation of the characteristics gives

$$S = -0.121 \pm 0.008,$$
$$E = +0.045 \pm 0.004.$$ [6]

Drawing the theoretic frequency curve from these values, one finds practically complete agreement with the observations.

68 The examples given shew that the form of the frequency curve is essentially connected with the number of elements in the given statistical series. A statistical object that for N, say, = 1000, gives rise to a nearly normal frequency curve, may give for a smaller N, say, 200, a frequency curve with notable skewness and excess. Obviously these values of skewness and excess, obtained from series with small values of N, express no essential property of the statistical object treated.

We realise from this the necessity of giving, when computing S and E, their mean error. Only by this means is it possible to decide whether the deviation of a given frequency curve from the normal form be accidental. We find from (3) that the mean errors of S and E (if the 'true' frequency curve differ little from the normal) depend only upon N and therefore are easily computed. As a practical rule (to which, however, important exceptions occur) it may be enunciated that it is not worth while in general to compute the higher characteristics of a statistical series, unless the number N of elements be greater than 1000. It is, however, always the mean error that tells us how far the values of skewness or excess are trustworthy.

69 A question which may be regarded as scarcely yet opened is that of how to explain departures of a frequency curve from the normal form. It is obvious that the interpretation is fundamentally connected with the nature of elementary sources of error. The difficulty, however, lies in that the solution of the problem is not unique, that even infinitely many solutions exist. Therefore it is necessary to classify these infinitely many solutions in a rational manner; but such a classification has not yet been attempted.

If, retaining the terminology of QUETELET, we call a statistical object characterised by a normal frequency curve a type, it can in general be said that a statistical object with non-normal frequency curve (with definite skewness, excess, etc.) may be considered composed of several types.

Assume for example that the height of 14-year-old schoolboys in Sweden forms a certain type and that 15-year-old schoolboys form another. Should a statistical object be now constructed by choosing at random 1000 15- and 1000 14-year-old schoolboys, the frequency curve for these 2000 individuals would no longer be normal. To be sure the skewness would be zero, but the curve would

6. <E_E = 0.033. --Tr.>

69 possess a negative excess. If the number of individuals from the two classes were unequal, a more or less notable skewness would also result.

Or, to take another example, assume that the 1000 14-year-old schoolboys had the same mean height as 1000 15-year-old schoolgirls chosen at random. Should one investigate the frequency curve for these 2000 individuals, one would find that this curve possessed positive excess and zero skewness.

Each statistical object that is a combination of a rather large number of types will thus in general possess a frequency curve of other than normal form. It will be understood from this that the normal curve (the Gaussian curve) must be not the rule, but the exception, in statistics. If, however, frequency curves with a large number of elements appear to depart only insignificantly from the normal form, it can by no means be concluded that the statistical objects consist of only a small number of types. Rather must we in general explain this behaviour by the fact that the number of types is very large.

If for any reason we know that a certain frequency curve consists of the sum of two normal curves (types), we can then compute from the characteristics of the given frequency curve the mean and dispersion of these subordinate types The solution of this problem was given by PEARSON [17, No. 2]. In exceptional cases the solution can also be found when the statistical object is compounded of three or more types.

Cap. XII. Frequency Curves of Type B.

70 Whenever the composition of a frequency curve can be explained by a summation of elementary errors, the resulting frequency curve must belong either to the 'Type A' discussed in the preceding chapter, or to that other form which I have named the frequency curve of Type B and which I shall briefly consider in this chapter.

The equation for a frequency curve of Type B is given, like that of Type A, by an infinite series of terms. The first of these was already exhibited by POISSON, and was later applied by BORTKIEWICZ to certain statistical problems. The general form of the equation of these curves, derived from the hypothesis of elementary errors, was first given by me in my paper 'Uber die zweite Form des Fehlergestzes' [3].

For the genesis of these curves I must refer to my more complete account. Let it only be mentioned that Type B occurs principally when the attribute is discontinuously bounded at one

of its values, so that on one side of this bound no individuals possessing the attribute occur, whereas on the other side of the bound and in its immediate neighbourhood the property occurs in a large number of individuals of the population.

If, for example, one wished to investigate the frequency curve for the number of voters at Swedish elections under the old conditions, by which no-one with an income under 800 Kronen had a vote, one would obviously have an attribute (the right to vote) that was discontinuously bounded.

There were no voters with incomes under 800 Kronen, but immediately over this figure, *ex.gr.* from 800 to 900 Kronen, there were many voters, who, because of this condition, belonged to Type B, when arranged according to income.

It is well to observe that the two types are not sharply distinguished from each other, but that, in general, transition forms exist, which, at least in practice, may be computed *ad lib.* as of one or the other type indifferently.

71 In homograde statistics the size of the element is always discontinuously bounded by the value zero. There can exist no element to which belongs less than no individual. Therefore, in general, homograde series of elements comprising a very small number of individuals have frequency curves of Type B. Thus, with respect to homograde statistical series, Type B may be named the frequency curve of rare events.

For example, the frequency curve for the number of stormy days in a month in Sweden (*v.* Art. 73 *infra*) is of Type B, whereas the number of rainy days, which are not so rare as stormy days, yields a frequency curve of Type A.

72 The series that represents the equation of a frequency curve of Type A has, as we saw in Art. 62 , the derivates of the normal function φ_0 as terms. In like manner the equation of a frequency curve of Type B may be represented as a series, which, however, has as terms the differences of a function $\psi(x)$. This function is given by the formula

$$\psi(x) = \frac{e^{-\lambda} \lambda^{x}}{\lfloor x} \tag{1}$$

where λ is the first characteristic of the statistical series. It can be chosen in different ways; I shall discuss only one here: that λ coincides with the mean of the x-values. We have then[1]

$$\lambda = M(x) = \Sigma x F(x) : N. \tag{2}$$

1. Since this determination of λ is not necessary, I have given a proper name to the characteristic λ: the modulus of the statistical series.

72 If, as usual, we designate by $F(x)$ the number of individuals in the class x, we have

$$(3) \qquad F(x) = N\left[\psi(x) + \gamma_2 \Delta^2\psi + \gamma_3 \Delta^3\psi + \gamma_4 \Delta^4\psi + \ldots\right],$$

where N denotes the whole number of individuals and γ_2, γ_3, γ_4, etc., certain constants, which here, together with λ, are the characteristics of the statistical series. $\Delta^2\psi$, $\Delta^3\psi$, etc., denote the finite differences of ψ, calculated from the formula

$$(4) \qquad \Delta\psi = \psi(x) - \psi(x-1)$$

I must omit here the formulae for the calculation of γ_3, γ_4, etc., and exhibit only the expression for γ_2 (a quantity I call the eccentricity) because its value is generally required in practical applications. We have

$$(5) \qquad \gamma_2 = \Sigma x^2 F(x) : 2N - \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda.$$

The dispersion (σ) of a series of Type B can be computed from the formula

$$(6) \qquad \sigma^2 = \lambda + 2\gamma_2$$

which can also be applied to compute γ_2 when λ and σ are known.

73 The value of the function $\psi(x)$ can easily be computed from the equation (1).[2] The following diagram gives a graphic representation of the function for different values of λ.

By means of this figure we can form a picture of the appearance of frequency curves of Type B. For small values of the modulus λ the frequency is greatest for $x = 0$ and becomes smaller the larger x gets. If the modulus $\lambda = 1$, the frequency is as large for $x = 0$ as for $x = 1$. For larger values of λ the frequency first increases and reaches a maximum for $x = \lambda - 1$, and then (for $x > \lambda$) decreases. For large values of λ the curves approach those of Type A.[3]

74 I shall confine myself to a single example for frequency curves of Type B.

The number of stormy days in Lund in different months during the years 1753-1857 is given in the following table.

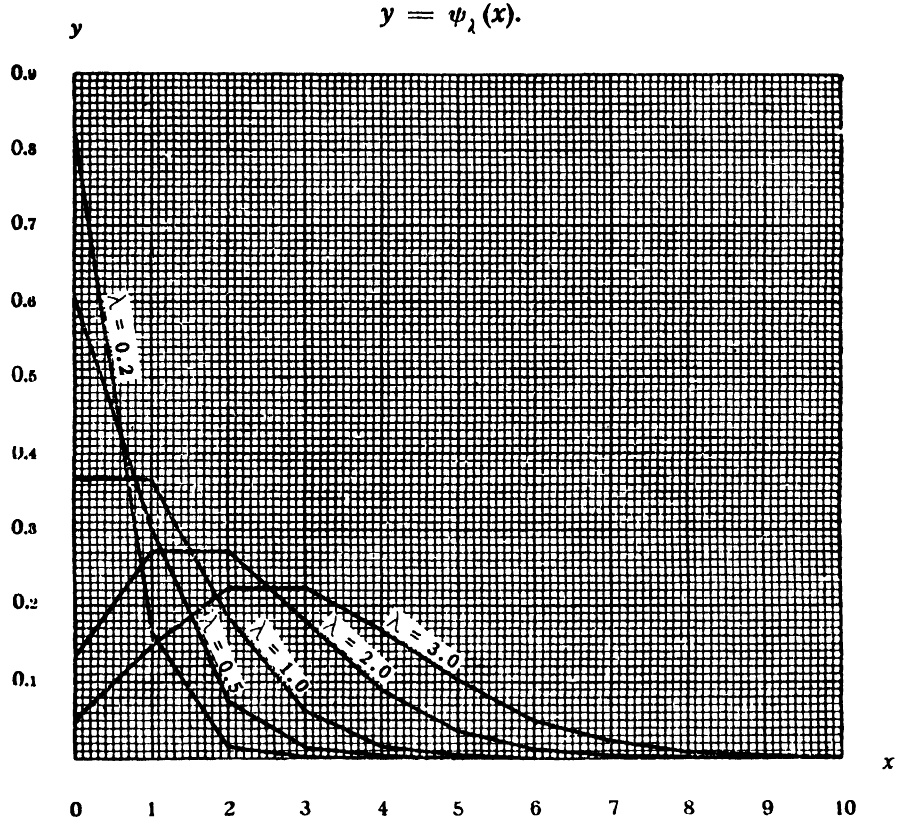
2. <A short table of $\psi(x)$ is given in Appendix Table 43. --Tr.>

3. Formula (3) presupposes that x takes on the values 0, +1, +2, etc. The value $x = 0$ corresponds to the bounding value of the property, so that there are no individuals to be found corresponding to negative values of x. It must be noted that $\psi(x)$ vanishes for all integral negative values of x.

Frequency curves of Type B.

Fig. 6.

$$y = \psi_\lambda(x).$$



This table is to indicate that in the 105 years here considered, no storms occurred in January, February, and December. In March there was no storm in 101 of the 105 years and in four years one stormy day, etc.

The values of the characteristics, computed by (2) and (5), are to be seen in the adjacent table.

Here λ gives the average number of stormy days during a month. The influence of the eccentricity γ_2 in the later summer months is significant. In order to give a picture of the influence of the second term in (3) for large values of γ_2, I adduce here the result for the month of August, compared with the observations.

Table 28.

Number (x) of stormy days in Lund in the years 1753-1857 .

$N = 105.$

x	Jan.	Feb.	March	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	x
0	105	105	101	82	48	27	12	24	62	93	102	105	0
1			4	17	36	36	25	26	24	11	3		1
2				6	14	26	24	19	9	1			2
3					5	9	15	13	5				3
4					2	5	14	9	5				4
5						2	9	6					5
6							2	5					6
7							1	2					7
8							1	0					8
9							1	0					9
10							1	0					10
11								1					11

Table 29.

Month	λ	r_2
January....	0.00	+ 0.000
February...	0.00	+ 0.000
March......	0.04	+ 0.000
April.......	0.23	+ 0.020
May	0.88	+ 0.047
June.........	1.38	+ 0.027
July.........	2.52	+ 0.781
August......	2.18	+ 1.002
September.	0.78	+ 0.246
October....	0.12	+ 0.002
November...	0.08	+ 0.000
December..	0.00	+ 0.000

The correspondence between the theoretic values (in the paenultimate column) and the observed values is very good. The first term in formula (3), taken alone, would here have given a very incomplete picture of the distribution of stormy days, as is seen from comparison of the fifth column with the last.

Table 30.

Number of stormy days in August.

$\lambda = 2.188, \gamma_2 = +1.002.$

x	$\psi(x)$	$\Delta\psi$	$\Delta^2\psi$	$N\psi$	$N\gamma_2\Delta^2\psi$	F	Frequency observed
0	+0.118	+0.118	+0.118	+12.4	+12.4	+24.8	24
1	+0.252	+0.184	+0.016	+26.5	+1.7	+28.2	26
2	+0.269	+0.017	−0.117	+28.2	−12.3	+15.9	19
3	+0.191	−0.078	−0.095	+20.1	−9.9	+10.2	13
4	+0.102	−0.089	−0.011	+10.7	−1.2	+9.5	9
5	+0.044	−0.058	+0.031	+4.6	+3.3	+7.9	6
6	+0.015	−0.029	+0.029	+1.6	+3.0	+4.6	5
7	+0.005	−0.010	+0.019	+0.5	+2.0	+2.5	2
8	+0.001	−0.004	+0.006	+0.1	+0.6	+0.7	0
9	+0.000	−0.001	+0.008	+0.0	+0.8	+0.8	0
10	+0.000	+0.000	+0.001	+0.0	+0.1	+0.1	0
11	+0.000	+0.000	+0.000	+0.0	+0.0	+0.0	1

Cap. XIII. On Correlation.

75 One of the most important tasks of practical statistics is the attempt to determine whether, and in what degree, different statistical events are dependent upon one another. The problem is as a rule very difficult because of 'accidental' fluctuations in the elements of the statistical series, which more or less strongly mask the dependency of the series.

For a long time the graphic method has been used to investigate the connexion between statistical series. One draws a curve that shews how the elements of a statistical series vary, and on the same paper another curve that portrays the variation of the elements of a second simultaneously observed statistical series. Then, by comparative observation of these curves, one can, at least in many cases, decide whether there is a connexion between the events.

The method is admirable and very necessary for the first recognition of connexion. It possesses, however, two important faults: firstly the accidental errors may be so large that it is difficult or even impossible to decide by graphic comparison whether a connexion between the events exist; secondly--and this is the most notable shortcoming of the method--the degree of the connexion between the events cannot be determined in this way.

75 In the great upswing of mathematical statistics that began at the end of the last century, attention was also turned to this problem, leading to the discovery of new and more reliable methods of determining the connexion between statistical events. The fundamental discovery in this field was made by Sir FRANCIS GALTON and is set forth in his work, notable also in other respects, *Natural Inheritance* [10].

76 The connexion between statistical events is denoted by the expression correlation. From the standpoint of elementary errors, two events are called correlated if they, wholly or partly, are due to the same elementary errors.

Strictly speaking it can obviously be said that all events in the world are correlated with one another. In most cases, however, the correlation is so weak that in practice the events may be considered independent. The strength of correlation is measured by the coefficient of correlation.[1] I shall discuss in the sequel the numerical computation of this coefficient and at the same time present some of its most important properties.

77 The method of computation is somewhat different, accordingly as the material is or is not arranged in classes. The latter case, comprising series of small extent, will be treated in this chapter.

We assume that two simultaneously observed statistical series S_1 and S_2, with the elements

$$m_1, m_2, m_3, \ldots, m_N,$$
$$n_1, n_2, n_3, \ldots, n_N,$$

are at hand, where the elements are supposed to correspond by pairs (m_1 and n_1, m_2 and n_2, etc.). Let the mean and dispersion of S_1 be M_1 and σ_1; those of S_2, M_2 and σ_2. To compute these characteristics as well as to compute the coefficient of correlation a provisional mean is chosen for each series, and denoted by M_{10} and M_{20}. Let r denote the coefficient of correlation. Putting

$$(1) \qquad x_k = m_k - M_{10}, \; y_k = n_k - M_{20},$$

where $k = 1, 2, 3, \ldots, N$, we have the following formulae for the computation of M_1, σ_1, M_2, σ_2, and r:

$$(2) \qquad b_1 = \Sigma x_k : N, \qquad b_2 = \Sigma y_k : N,$$

1. More generally, the correlation is given by an infinite series of characteristics, of which the first--and in general most important--is the coefficient of correlation. I cannot go into the higher characteristics of correlation in this resume. <See JØRGENSEN <12>. --Tr.>

(3) $$M_1 = M_{10} + b_1, \qquad M_2 = M_{20} + b_2,$$

(4) $$\sigma_1^2 = \Sigma x_k^2 : N - b_1^2, \quad \sigma_2^2 = \Sigma y_k^2 : N - b_2^2,$$

(5) $$\sigma_1 \sigma_2 r = \Sigma x_k y_k : N - b_1 b_2.$$

Formulae (1) to (4) are nothing but the already given formulae for computing the means and dispersions of the given series. The coefficient of correlation r is furnished by formula (5).

I shall apply the formulae to a numerical example and shew how a simple check on the computation may simultaneously be imposed.

78 The supply of drinking water in the city of Lund comes in part from five reservoirs in Rögle, 5 km from the city. Some time ago I was engaged by the city to investigate in what degree the water supply might be dependent upon meteorologic factors. I had, therefore, as first task, to determine the correlation between the quantity of water that flowed into the reservoir in one year[2] and the total rainfall during the same time.

The investigation is based on the following figures, which were obtained in the years 1899-1908 . Let

T = the whole inflow of water to the reservoir in a year, expressed in 1000 cbm,

R = the total rainfall during the same time, expressed in mm.

A glance at this table suffices to convince us that a connexion exists between the rainfall and the inflow of water. In dry years (1899 , 1902 , 1904) the flow into the reservoirs is small, likewise in years with much rain the inflow is in general large. The

Table 31.

Year	T	R
1899	258	511
1900	708	661
1901	426	597
1902	304	541
1903	762	663
1904	266	563
1905	562	607
1906	422	576
1907	521	530
1908	522	719

2. It is necessary to extend the investigation to the different months of the year. For the sake of brevity I restrict myself to the results fo- different years.

78 fluctuations, however, are not insignificant (*cf.ex.gr.* the years 1907 and 1908), and it is necessary to determine to what extent the two events are mutually dependent.

We take the provisional mean at 500 for T and 600 for R, so that

$$T = 500 + x,\ R = 600 + y.$$

Then the computation of the characteristics proceeds according to the schema of Table 32 .

Table 32.

Correlation between rainfall and influx of water at Rogle.

$$N = 10,\ M_{10} = 500,\ M_{20} = 600.$$

Year	x	y	xx	xy	yy	$x+y$	$(x+y)^2$
1899	−242	−89	+58 600	+21 500	+7 900	−331	109 600
1900	+208	+61	+43 300	+12 700	+3 700	+269	72 400
1901	−74	−3	+5 500	+200	0	−77	5 900
1902	−196	−59	+38 400	+11 600	+3 500	−255	65 000
1903	+262	+63	+68 600	+16 500	+4 000	+325	105 600
1904	−234	−37	+54 800	+8 700	+1 400	−271	73 400
1905	+62	+7	+3 800	+400	0	+69	4 800
1906	−78	−24	+6 100	+1 900	+600	−102	10 400
1907	+21	−70	+400	−1 500	+4 900	−49	2 400
1908	+22	+119	+500	+2 600	+14 200	+141	19 900
	−249	−32	+280 000	+74 600	+40 200	−281	469 400

The last two columns have been added as a check. For we have

$$\Sigma(x+y)^2 = \Sigma x^2 + \Sigma y^2 + 2\,\Sigma xy.$$

The table gives

$$\begin{aligned} \Sigma x^2 &= +280\,000 \\ \Sigma y^2 &= +\ \ 40\,200 \\ 2\,\Sigma xy &= +149\,200 \\ \hline & +469\,400 = \Sigma(x+y)^2, \end{aligned}$$

whereby the entire computation is completely checked.

From the numbers in the bottom line we now have

$$b_1 = -249:10 = -24.9; \qquad b_2 = -32:10 = -3.2,$$

$$\sigma_1^2 = 280\,000 : 10 - b_1^2 = 27\,380; \quad \sigma_2^2 = 40\,200 : 10 - b_2^2 = 4010,$$

so that

$$\sigma_1 = 165.5, \quad \sigma_2 = 63.3.$$

Further we obtain from (5)

$$r\,\sigma_1\,\sigma_2 = 74\,600 : 10 - b_1\,b_2 = +7380,$$

so that, with the values obtained for σ_1 and σ_2,

$$r = +0.704$$

is the coefficient of correlation.

79 What need prompts the computation of the coefficient of correlation? To explain this, we shall first elucidate some general properties of the coefficient.

Mathematical analysis shews that the numerical value of r is less than or equal to unity. Moreover, the coefficient of correlation may be positive or negative; if it is positive, then, in general, the two attributes considered increase together and decrease together. If r is negative, then one attribute increases when the other decreases, and *vice versa*. In the example considered r was positive, and consequently the inflow of water to the reservoirs is in general greater in rainy years than in dry. This is also directly seen from Table 31 .

A coefficient of correlation of zero signifies that the observed statistical events are mutually independent and have nothing to do with each other.

If $r = +1$ (or -1), an element in one statistical series is completely determined by the corresponding element in the other series.

It is, then, to be understood that the larger the numerical value of r, the closer the dependency between the observed statistical series. For example r has been found $= +0.96$ from simultaneous measurements of the right and left femur, $= +0.80$ between the height and the length of femur, and only $= +0.37$ between the height and the length of forearm. The correlation r between the temperature in Lund in June and July $= +0.734$; that between June and November $= +0.093$.

80 The coefficient of correlation is not limited to furnishing a measure of the intensity of the connexion between statistical events. It also serves to solve the following important problem:

Given the value of an element of one series, what is the most probable value of the corresponding element in the other series?

80 Let x designate the deviation of an element in the first series from its mean M_1; let y designate the deviation of the corresponding element in the second series from its mean M_2. Denoting by X the most probable value of x corresponding to a given y, we have

$$X = r\frac{\sigma_1}{\sigma_2}y. \tag{6}$$

Contrariwise, denoting by Y the most probable value of y corresponding to a given x, we have

$$Y = r\frac{\sigma_2}{\sigma_1}x. \tag{7}$$

Equations (6) and (7) represent geometrically two straight lines, known as the regression lines. The coefficients of y and of x in the right sides of equations (6) and (7) are called the regression coefficients.

If $\sigma_1 = \sigma_2$, these equations assume the form $X = ry$ and $Y = rx$. Since r is always numerically smaller than unity, it follows from the above equations that in this case the most probable value of an element of the second series departs less from its mean than the given element departs from the mean of the first series. Hence the name *regression*.

81 We obtain as the equations for the regression lines in the example of Art. 78 ,

$$T - 375 = 1.84\ (R - 597),$$
$$R - 597 = 0.27\ (T - 375),$$

where the T and R on the left side denote the most probable values of inflow and rainfall corresponding respectively to given values of R and T. The first of these equations can be used to compute the most probable value of flow of water into the reservoirs from knowledge of the fluctuations of rainfall in Lund.

82 As with other characteristics, it is again very necessary to calcualte the mean error of the coefficient of correlation; this is done by means of the formula

$$\varepsilon(r) = \frac{1 - r^2}{\sqrt{N}}. \tag{8}$$

As in many other cases the mean error varies inversely as the square root of the number of observations. For constant N it is smaller, the larger r is. In our example we have

$$\varepsilon(r) = \frac{1 - (0.704)^2}{\sqrt{10}} = 0.157,$$

so that the value of r is to be written in the form

$$r = +0.704 \pm 0.157.$$

83 As a suitable example to shew the great practical utility of the mathematical theory of correlation, I shall give the results of an investigation of the correlation between the temperatures in Lund for different months. The numbers are derived from observations in Lund over the Years 1753-1857

Table 33.

Correlation between the temperatures in different months in Lund.

$N = 105.$

M	σ	Mon.	Jan.	Feb.	March	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
$-2^0.21$	$2^0.57$	Jan.												
-2.19	2.84	Feb.	$+0.330$											
-1.19	2.68	March	$+0.284$	$+0.580$										
$+3.04$	1.80	Apr.	$+0.182$	$+0.283$	$+0.421$									
$+8.42$	1.99	May	$+0.181$	$+0.282$	$+0.303$	$+0.429$								
$+12.99$	1.99	June	$+0.258$	$+0.188$	$+0.209$	$+0.409$	$+0.586$							
$+15.10$	1.91	July	$+0.241$	$+0.059$	$+0.146$	$+0.361$	$+0.518$	$+0.784$						
$+14.65$	1.77	Aug.	$+0.184$	$+0.284$	$+0.178$	$+0.306$	$+0.498$	$+0.589$	$+0.695$					
$+11.46$	1.52	Sept.	$+0.302$	$+0.240$	$+0.154$	$+0.184$	$+0.384$	$+0.449$	$+0.401$	$+0.520$				
$+7.27$	1.76	Oct.	$+0.057$	$+0.109$	$+0.157$	$+0.221$	$+0.268$	$+0.271$	$+0.128$	$+0.161$	$+0.256$			
$+2.89$	1.78	Nov.	-0.098	-0.110	$+0.015$	$+0.128$	$+0.147$	$+0.098$	$+0.062$	$+0.041$	$+0.141$	$+0.104$		
-0.19	2.81	Dec.	-0.156	$+0.181$	$+0.070$	$+0.109$	$+0.281$	$+0.189$	$+0.148$	$+0.281$	$+0.104$	$+0.128$	$+0.242$	

Unfortunately I can not here go into a further discussion of this interesting series of correlations.

Cap. XIV.

On Correlation between Series arranged in Classes.

84 If the number of elements is so large that a division into classes is indicated, the coefficient of correlation, like the other characteristics, can be computed with greater accuracy. The computational process is significantly abridged by division into classes. The form of computation is changed, although formulae (1) to (7) of the previous chapter remain valid. I shall carry out an explicit example of correlation between classified series, and emphasise the points that may be of interest in the treatment of such a problem.

Table 34.

$N = 330$, $w_1 = 4$ cm, $w_2 = 10$ cm.

		Length of the topmost branch							Σ
	cm	1–4	5–8	9–12	13–16	17–20	21–24	25–28	
Height of the firs	1–10	5	3						8
	11–20	7	15	1	2				25
	21–30	2	21	17	2				42
	31–40		15	37	20	3			75
	41–50		1	31	26	4			62
	51–60		2	8	27	12	1		50
	61–70			2	9	24	4	2	41
	71–80			1	4	7	6	2	20
	81–90				1		3		4
	91–100						1	2	3
	Σ	14	57	97	91	50	15	6	330

85 Correlation between the height of fir trees and the length of their topmost branch.

In the summer of 1909 my daughters ESSIE and SONJA measured 330 young firs (3 to 4 years old) growing on Kämpinge-Heide in Schonen, and likewise the length of their topmost branch. The trees were all shorter than 1 m; the branches all shorter than 28 cm.

The values obtained in these 330 pairs of measurements are not given here; instead they are collected into a so-called correlation table. The heights of the firs are grouped in classes with class interval (w_2) of 10 cm, and the corresponding lengths of branches in classes with class interval (w_1) of 4 cm. Table 34 gives a conspectus of the measurements.

It is seen from this table that five firs had a height of 1 to 10 cm and a top of 1 to 4 cm in length. It is further seen that three firs, shorter than 10 cm, had tops of 5 to 8 cm in length, etc.

For numerical computation, provisional means are first chosen for the two lengths. We take the class 13 to 16 cm as provisional mean for the tops. The centre of this class, M_{10}, is 14.5 cm. For the height of the firs we choose the centre of the class 41 to 50 cm as provisional mean; M_{20} = 45.5 cm. We number the classes of tops in order with the class marks -3 , -2 , -1 , 0 , +1 , +2 , +3 , so that the class 0 is that class containing the provisional mean M_{10}. We designate any of these class marks by x. Similarly we number the classes of height of firs and designate these marks by y. The computation of the characteristics now proceeds as in Table 35 .

We may call each small quadrangle, characterised by a certain x and a certain y, a subclass. It is seen that in each of these subclasses three numbers are entered. The number in large type, designated F (= frequency), gives the number of individuals within each subclass, and is the same as the corresponding number in Table 34 . Then the products of F by the class marks x and y are entered in small type. The numbers in the same column have the same x-value; the numbers in the same row have the same y-value. When the number of classes is not large, these products are obtained without the help of a computing machine. After all subclasses have been filled up in this manner, all numbers are added horizontally and vertically. Their sums are then multiplied, as the table shews, by x and y. Finally all numbers of the last row and last column are summed, and the sums entered in the lower right corner. The seven sums obtained in this manner give the desired characteristics, as shall immediately be demonstrated.

Noticing that the numbers F indicate how often each combination of x and y occurs in the population at hand, we may directly

Table 35.

Computation of the coefficient of correlation.

$N = 330$, $M_{10} = 14.5$, $M_{x0} = 45.5$, $w_1 = 4$ cm, $w_2 = 10$ cm.

$x =$ / y	−3 F, xF yF	−2 F, xF yF	−1 F, xF yF	0 F, xF yF	+1 F, xF yF	+2 F, xF yF	+3 F, xF yF	ΣF, ΣxF, $y\Sigma xF$ ΣyF, $y\Sigma yF$
−4	5 −15 −20	3 −6 −12						8 −21 +84 −32 +128
−3	7 −21 −21	15 −30 −45	1 −1 −3	2 0 −6				25 −52 +156 −75 +225
−2	2 −6 −4	21 −42 −42	17 −17 −34	2 0 −4				42 −65 +130 −84 +168
−1		15 −30 −15	37 −37 −37	20 0 −20	3 +3 −3			75 −64 +64 −75 +75
0		1 −2 0	31 −31 0	26 0 0	4 +4 0			62 −29 0 0 0
+1		2 −4 +2	8 −8 +8	27 0 +27	12 +12 +12	1 +2 +1		50 +2 +2 +50 +50
+2			2 −2 +4	9 0 +18	24 +24 +48	4 +8 +8	2 +6 +4	41 +38 +72 +82 +164
+3			1 −1 +3	4 0 +12	7 +7 +21	6 +12 +18	2 +6 +6	20 +24 +72 +60 +180
+4				1 0 +4		3 +6 +12		4 +6 +24 +16 +64
+5						1 +2 +5	2 +6 +10	3 +8 +40 +15 +75
ΣF, ΣxF ΣyF	14 −45 −42	57 −114 −112	97 −97 −59	91 0 +31	50 +50 +78	15 +30 +44	6 +18 +20	330 −155 +644 −43 +1129
$x\Sigma xF$ $x\Sigma yF$	+126 +135	+228 +224	+97 +59	0 0	+50 +78	+60 +88	+54 +60	+615 +644

apply formulae (2) to (5) of the previous chapter and obtain

$$b_1 = -155 : 330 = -0.470, \quad b_2 = -43 : 330 = -0.130,$$

$$\sigma_1 = \sqrt{615 : 330 - b_1^2} = +1.283, \quad \sigma_2 = \sqrt{1129 : 330 - b_2^2} = +1.845,$$

$$\sigma_1 \sigma_2 r = 644 : 330 - b_1 b_2 = +1.891,$$

$$r = +0.825.$$

Here b_1, b_2, σ_1, and σ_2 are expressed in class interval units. r is an abstract number and therefore independent of the units of the attributes observed. If it is desired to express the characteristics in centimeters, b_1 and σ_1 must be multiplied by w_1 (= 4 cm); b_2 and σ_2 by w_2 (= 10 cm). Adding $b_1 w_1$ to M_{10}, $b_2 w_2$ to M_{20}, we obtain the means M_1 and M_2 for the height of the firs and the length of the tops. Carrying out this work, and adding the mean errors of M and σ from formulae (2) and (3) of Cap. IV, and the mean error of r from formula (8) of Cap. XIII, we have the values

$$M_1 = 12.62 \text{ cm} \pm 0.28; \quad M_2 = 44.20 \text{ cm} \pm 1.02,$$
$$\sigma_1 = 5.13 \text{ cm} \pm 0.20; \quad \sigma_2 = 18.45 \text{ cm} \pm 0.72,$$
$$r = +0.825 \pm 0.018.$$

86 I now pass to consideration of the regression lines and return for this purpose to Table 34 . As already mentioned, this arrangement is known as a correlation table. A column or row is called an array by the English writers; I shall retain this name. More precisely a vertical row is called a y-array and a horizontal row an x-array. That value of x or of y that is common to all the elements of the same array is ordinarily called its type. Thus the fourth horizontal row of Table 34 is an x-array of type 35.5 ; the first vertical row a y-array of type 2.5 .

Each array is a statistical series and thus possesses a mean and a dispersion. The following interesting theorems hold for these characteristics:

The means of the arrays lie on the regression lines.

The dispersion of all x-arrays is the same and has the value $\sigma_1\sqrt{1-r^2}$; likewise all y-arrays have equal dispersion of value $\sigma_2\sqrt{1-r^2}$.

87 Let us now see how far these theorems are fulfilled in the present example.

To simplify the treatment, the values of x and y will be taken in class interval units. We may then employ Table 35 directly. The equations for the regression lines, according to formulae (6) and (7) of the preceding chapter, are:

$$(1) \qquad X - b_1 = r\frac{\sigma_1}{\sigma_2}(y - b_2),$$

$$(2) \qquad Y - b_2 = r\frac{\sigma_2}{\sigma_1}(x - b_1),$$

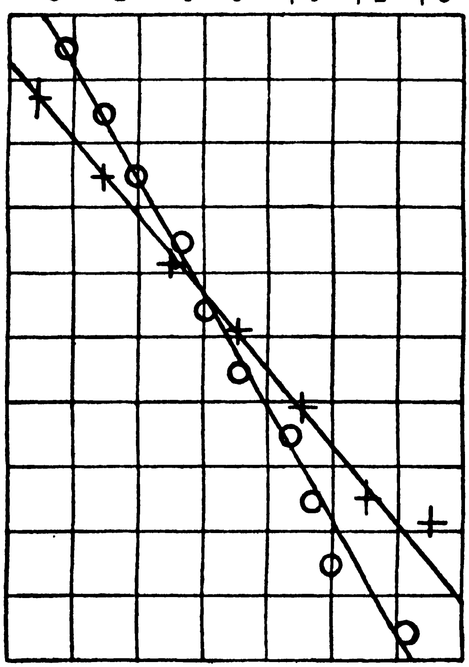
87 where, as before, X signifies the most probable value of x corresponding to a given y, and Y the most probable value of y corresponding to a given x. Substituting the numbers from Art. 85 these equations become

$$\text{for given } y\text{: } X + 0.47 = 0.573\,(y + 0.13);$$
$$\text{for given } x\text{: } Y + 0.13 = 1.188\,(x + 0.47).$$

These two regression lines are drawn in the following diagram (Table 36), upon a system of coordinates whose origin is in the point $x = y = 0$.

Table 36.

Regression lines.



	−3	−2	−1	0	+1	+2	+3	Σ	Mean	Dispersion
−4								8	−2.62	0.51
−3								25	−2.08	0.79
−2								42	−1.55	0.66
−1								75	−0.85	0.78
0								62	−0.47	0.64
+1								50	+0.04	0.80
+2								41	+0.88	0.81
+3								20	+1.20	1.08
+4								4	+1.50	0.67
+5								3	+2.67	0.45
Σ	14	57	97	91	50	15	6	330		
Mean	−3.21	−1.97	−0.61	+0.34	+1.56	+2.98	+3.38			
Disp.	0.70	1.07	1.08	1.39	1,08	1.01	1.26			

The means of the different x- and y-arrays are also drawn on this system of coordinates. The means of the x-arrays are indicated by small circles, the means of the y-arrays by crosses.

It is seen that the means lie very near the regression lines. The departures are in complete accord with the corresponding mean errors of the means; the calculation of these mean errors is here omitted.

It must be remarked that the two lines of regression (1) and (2) would coincide if r were $= 1$ and the correlation were perfect. This follows directly from equations (1) and (2). In the present case the coefficient of correlation is reasonably large, so that the angle between the lines of regression is insignificant.

Considering, further, the dispersions of the different arrays, the numbers in the last vertical row of Table 36 , which gave the values of dispersion in the x-arrays, shew that these values are indeed not identical, but variable like any other statistical event. The fluctuations are again in accord with the mean error, and taking the mean of all values of dispersion, we have a number

$$0.73$$

that accords well with the theoretic value

$$\sigma_1 \sqrt{1-r^2} = 0.728.$$

Similar remarks hold for the dispersions of the y-arrays. The mean of the dispersions observed is

$$1.06,$$

and the theoretic value

$$\sigma_2 \sqrt{1-r^2} = 1.043.$$

The regression line (1) enables us to compute the most probable value of the length of the top of a fir of given height. From the line (2) we can compute the most probable height of a fir whose top has a given length. This is not the place to investigate the degree to which these questions may be of biologic interest.

88 Whenever one must investigate whether and to what extent two statistical events are mutually dependent, one must turn to the theory of correlation. The solution, however, is not always so easy to find as we have assumed in these chapters. Indeed the coefficient of correlation r gives in many cases all available in-

88 formation as to this connexion. If, however, the task is complex, or the number of individuals in the population very large, so that greater exactitude is practicable and desirable in the treatment of the problem, the higher characteristics of the correlation function must also be determined.

I shall not go more closely into this question here, but shall merely point out that the correlation function can in general be represented by an equation that is a direct generalisation of the equation we introduced in Cap. XI for frequency curves of Type A. The coefficients in this series may be obtained by elementary means analogous to those by which we obtain the skewness and excess of a statistical series.

89 The relation connecting a certain given value of one attribute with the most probable value of the other takes on a more complicated form than that of formulae (1) and (2) of Art. 87 when the higher characteristics are taken into account. The lines of regression which give this connexion are no longer right lines, but are in general curves resembling hyperbolae. If, in the example of this chapter, we had had also to consider measurements of firs of age 20 to 30 years (instead of only 3 to 4 years), it would have been necessary, to solve the problem of correlation, to take the higher characteristics of the correlation function into account.

Cap. XV.

Abridged Methods for calculating the Characteristics.

90 I have treated in previous chapters the general methods for computing the coefficient of correlation. In special cases, which, however, may be of great practical importance, these methods fail us. Particularly does this happen when, for some reason, the statistical material is divided into a very small number of classes. The methods of this chapter refer to this division.

91 To treat these problems we use certain functions that have not been needed previously: firstly the integral

$$Q(x) = \int_{\infty}^{x} e^{-\frac{t^2}{2}} dt, \quad (1)$$ [1]

and secondly the inverse of this function, denoted by $R(u)$. Thus

1. <In the tables, Q is designated ϕ_{-1}. --Tr.>

$$u = Q(x); \quad x = R(u). \tag{2}$$

For the shape of these functions see the tables at the end of this book. I merely remark here that

$$\begin{aligned} Q(-x) &= 1 - Q(x), \\ R(1-u) &= -R(u). \end{aligned} \tag{3}$$

92 Computation of the mean and dispersion from material divided into three classes.

Let some attribute of an object be under study, and let x denote the degree of the attribute. If, then, x_1 and x_2 are two values of the attribute, and we know the number (A_1) of individuals whose attribute is smaller than x_1, and the number (A_2), whose attribute is smaller than x_2, as well as the whole number (N) of individuals, we can compute the mean and dispersion of x.[2]

First we have

$$\begin{aligned} A_1 &= NQ(X_1), \\ A_2 &= NQ(X_2), \end{aligned} \tag{4}$$

where X_1 and X_2 are the normal coordinates corresponding to x_1 and x_2, so that

$$\begin{aligned} \frac{x_1 - M}{\sigma} &= X_1, \\ \frac{x_2 - M}{\sigma} &= X_2. \end{aligned} \tag{5}$$

But from (4) we have

$$\begin{aligned} X_1 &= R\left(\frac{A_1}{N}\right), \\ X_2 &= R\left(\frac{A_2}{N}\right), \end{aligned} \tag{6}$$

where R designates the function R of Art. 91 .

Since A_1/N and A_2/N are known, we can compute X_1 and X_2 from Table 42 at the end of the book. Equations (5) then give the values of M and σ, and so we have

2. If, as well as the mean and the dispersion, the total number of individuals is unknown, the problem can be solved similarly. I refer you, in this connexion, to <5>.

92

(7)
$$\sigma = \frac{x_1 - x_2}{X_1 - X_2},$$
$$M = x_1 - \sigma X_1.$$

In this process it is irrelevant whether the statistical series treated be homograde or heterograde.

93 Example. We return to the first example of the computation of the dispersion (p. 16). The full distribution, of the number of boys per 500 births, is given in Table 4 . Let us assume that we only knew that in 147 of the 576 cases investigated there were fewer than 249 boys in 500 , and in 531 cases fewer than 274 . Then

$$x_1 = 249, \quad x_2 = 274,$$
$$A_1 = 147, \quad A_2 = 531.$$

Further $N = 576$.

So we obtain $A_1/N = 0.255$, $A_2/N = 0.922$, and from Table 42 $X_1 = -0.659$, $X_2 = +1.419$, and hence

$$\sigma = \frac{274 - 249}{1.419 + 0.657} = \frac{25}{2.076} = 12.0$$
$$M = 249 + 12.04 \times 0.659 = 256.9,$$

in good agreement with the values found in Capp. II. and III.

94 As second example I take the following. In measuring the cephalic index of humans, anthropologists distinguish between dolichocephalics with an index smaller than 75 , brachycephalics with an index greater than 80 , and mesocephalics with an index between 75 and 80 . In the work of C. M. Furst and F. C. C. Hansen [9 , p. 121] is found the following synopsis of the distribution of dolichocephalics, etc., for different times and places.

Table 37.

Distribution of the cephalic index.

	Dolichocephaly <75	Mesocephaly $75-80$	Brachycephaly >80
Greenland	84 %	15 %	1 %
Sweden, stone age	51 %	40 %	9 %
" iron age	66 %	29 %	5 %
" modern age	30 %	57 %	13 %
Bavaria	1 %	16 %	83 %

We denote the cephalic index by x: in this case $x_1 = 75$, $x_2 = 80$. By the rules of Art. 92 we have

Table 38.

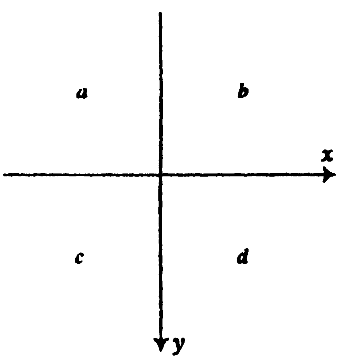
	$A_1:N$	$A_2:N$	X_1	X_2	σ	M
Greenland.............	0.84	0.99	$+0.99$	$+2.33$	3.7	71.3
Sweden, stone age....	0.51	0.91	$+0.08$	$+1.34$	3.8	74.9
„ iron age.....	0.66	0.95	$+0.41$	$+1.64$	4.1	73.8
„ modern age...	0.80	0.87	-0.52	$+1.18$	3.0	76.6
Bavaria................	0.01	0.17	-2.33	-0.95	3.6	83.4

It must however be noted that the mean error of a statistic is larger when computed by the abridged method than when computed by the full method.

95 Computation of coefficient of correlation from four subclasses in heterograde statistics.

In a collection of N individuals we consider two attributes, A and B, of the individuals. We denote the degree of the attribute A by x, that of the attribute B by y. Now we undertake a distribution of the N individuals into the following four subclasses:

Subclass a:	Individuals with	$x < x_1$;	$y < y_1$;
„ b:	„ „	$x > x_1$;	$y < y_1$;
„ c:	„ „	$x < x_1$;	$y > y_1$;
„ d:	„ „	$x > x_1$;	$y > y_1$.



This distribution may be seen in the adjacent figure. The intersection of the two right lines has here the coordinates $x = x_1$, $y = y_1$.

Obviously we have:

$$N = a + b + c + d$$

where a, b, c, d here designate the number of individuals in the corresponding subclass.

Designating the normal coordinates of the point of intersection by h and k, we have by formula (6):

95

$$h = R\left(\frac{a+c}{N}\right),$$

(8)

$$k = R\left(\frac{a+b}{N}\right).$$

Since a, b, c, d are known, we can compute the quantities h and k from (8). We cannot obtain the mean and dispersion of each of the two attributes; for this the knowledge of more subclasses were necessary. We can, however, compute the coefficient of correlation (r) of the two attributes. The solution of this problem, given by PEARSON, is the following. We introduce H and K, setting

$$H = \varphi_0(h); \quad K = \varphi_0(k); \tag{9}$$

a short table of φ_0 is given in Table 41 at the end. r is then obtained from the following equation:

$$\begin{aligned}\frac{ad-bc}{N^2 HK} &= r + \frac{r^2}{2} hk + \frac{r^3}{6}(h^2-1)(k^2-1) + \\ &+ \frac{r^4}{24} h(h^2-3)k(k^2-3) + \frac{r^5}{120}(h^4-6h^2+3)(k^4-6k^2+3) + \\ &+ \frac{r^6}{720} h(h^4-10h^2+15)k(k^4-10k^2+15) + \\ &+ \frac{r^7}{5040}(h^6-15h^4+45h^2-15)(k^6-15k^4+45k^2-15) + \\ &+ \dots\end{aligned} \tag{10}$$

If r is small, the root of this equation can easily be expanded in a series of powers of $(ad-bc)/N^2HK$. In other cases the root may be found by trial. PEARSON [19 , XXIX-XXX: 1-lv, 42-57 ; 20 , V-IX: xliv-lxxix, 73-109] has published various tables to facilitate this computation.

96 Computation of coefficient of correlation from four subclasses in homograde statistics.

We consider two attributes, A and B, of the individuals of a certain population. The individuals may then be distributed into four groups, which we shall denote by the symbols

$$AB, \; Ab, \; aB, \; ab$$

This symbolism is to signify, that those groups possessing the attribute A have the A in their symbols; those lacking it the a; likewise with B and b; so that we may consider a = non-A, b = non-B.

Taking an individual of the population at random, we shall designate the probability that this individual belong to a certain of these groups by p_1, p_2, p_3, p_4, respectively; so that p_1 is equal to the quotient of the number with the attributes A and B by the total number of the population.

Now let this process be performed not once, but s times (assuming that both s and the population P are large numbers); then, by a theorem known from probability, the probability $B(m_1, m_2, m_3, m_4)$ of obtaining in these s trials

m_1	Individuen	aus	der	Gruppe	AB,
m_2	„	„	„	„	Ab,
m_3	„	„	„	„	aB,
m_4	„	„	„	„	ab

is given by the following formula (where $s = m_1 + m_2 + m_3 + m_4$):

$$(11)\quad B(m_1, m_2, m_3, m_4) = \frac{\underline{|s}}{\underline{|m_1}\;\underline{|m_2}\;\underline{|m_3}\;\underline{|m_4}}\, p_1^{m_1} p_2^{m_2} p_3^{m_3} p_4^{m_4}.$$

This formula may be treated by the same method I have used in my paper 'Die strenge Form des BERNOULLIschen Theorems' [5] I confine myself here to the largest term of the solution. The formula of MOIVRE and STIRLING furnishes the approximations

$$(12)\quad \begin{aligned} \underline{|s} &= e^{-s} s^s \sqrt{2\pi s}, \\ \underline{|m_1} &= e^{-m_1} m_1^{m_1} \sqrt{2\pi m_1} \end{aligned}$$

etc., so that

$$(13)\quad B(m_1, m_2, m_3, m_4) =$$

$$= \frac{\left(\frac{m_1}{sp_1}\right)^{-(m_1+{}^1\!/_2)} \left(\frac{m_2}{sp_2}\right)^{-(m_2+{}^1\!/_2)} \left(\frac{m_3}{sp_3}\right)^{-(m_3+{}^1\!/_2)} \left(\frac{m_4}{sp_4}\right)^{-(m_4+{}^1\!/_2)}}{\sqrt{(2\pi)^3 s^3 p_1 p_2 p_3 p_4}}$$

The expression in the numerator is treated in the usual manner by setting

$$(14)\qquad \left(\frac{m_1}{sp_1}\right)^{-(m_1+1/2)} = e^{-(m_1+1/2)\log\frac{m_1}{sp_1}}$$

and expanding the exponents in series.

For this purpose four numbers l_1, l_2, l_3, and l_4 are introduced by the relations

$$(15)\qquad \begin{aligned} m_1 &= sp_1 + l_1, \\ m_2 &= sp_2 + l_2, \\ m_3 &= sp_3 + l_3, \\ m_4 &= sp_4 + l_4 \end{aligned}$$

so that (because $p_1 + p_2 + p_3 + p_4 = 1$)

$$(16)\qquad l_1 + l_2 + l_3 + l_4 = 0.$$

Expanding the exponent in (14) in powers of l_1, we have first

$$\log\frac{m_1}{sp_1} = \frac{l_1}{sp_1} - \frac{l_1^2}{2s^2p_1^2} + \frac{l_1^3}{3s^3p_1^3} - \dots,$$

whence we obtain, by keeping only the lowest powers of l_1,

$$(17)\quad (m_1+1/2)\log\frac{m_1}{sp_1} = l_1\left(1+\frac{1}{2sp_1}\right) + l_1^2\left(\frac{1}{2sp_1} - \frac{1}{4s^2p_1^2}\right) + l_1^3\left(-\frac{1}{6s^2p_1^2} + \frac{1}{6s^3p_1^3}\right) + \dots$$

Similarly developing the remaining factors in (13), the first term vanishes on account of (16). The largest of the remaining terms in (17) is

$$(18)\qquad \frac{l_1^2}{2sp_1}.$$

One might be inclined to believe that

$$(19)\qquad \frac{l_1}{2sp_1}$$

were of the same order of magnitude as (18), or even greater, and indeed this is the case for very small values of l_1. If, however, l_1 approximates sp_1, (18) is the most important term. The term

(19), together with the term in $l_1 3$, determines the 'skewness' of the curve, and will be neglected here.

Thus expression (13) becomes

$$(20) \qquad B(m_1, m_2, m_3, m_4) = \frac{e^{-\frac{l_1^2}{2sp_1} - \frac{l_2^2}{2sp_2} - \frac{l_3^2}{2sp_3} - \frac{l_4^2}{2sp_4}}}{\sqrt{(2\pi)^3 s^3 p_1 p_2 p_3 p_4}}.$$

Formula (20) is also applicable to correlation between mutually exclusive attributes. First, however, we have to solve the following correlation problem:

What is the probability of obtaining, in s trials, t_1 individuals with the attribute A and t_2 with the attribute B?

Obviously the t_1 individuals with the attribute A comprise first the m_1 individuals of the group AB and second the m_2 individuals of the group Ab. Likewise the t_2 individuals with the attribute B comprise the m_1 individuals of the group AB and the m_3 individuals of the group aB. Thus

$$t_1 = m_1 + m_2,$$

$$t_2 = m_1 + m_3.$$

Moreover,

$$s = m_1 + m_2 + m_3 + m_4,$$

so that the problem leaves one of the numbers m_1, m_2, m_3, m_4 undetermined. Let m_1 be this number. We can then express m_1, m_2, m_3, and m_4 in terms of t_1, t_2, s, and m_1; this gives the relations

$$(21) \qquad \begin{aligned} m_1 &= m_1, \\ m_2 &= t_1 - m_1, \\ m_3 &= t_2 - m_1, \\ m_4 &= s + m_1 - t_1 - t_2. \end{aligned}$$

It is advantageous to put

$$(22) \qquad \begin{aligned} t_1 &= sp_1 + sp_2 + \lambda_1, \\ t_2 &= sp_1 + sp_3 + \lambda_2, \end{aligned}$$

96 wherein λ_1, λ_2, are to be considered small numbers. Then we obtain instead of (21)

$$
(23)\qquad
\begin{aligned}
l_1 &= l_1, \\
l_2 &= \lambda_1 - l_1, \\
l_3 &= \lambda_2 - l_1, \\
l_4 &= l_1 - \lambda_1 - \lambda_2.
\end{aligned}
$$

If, inserting these values in (20), and expressing the right side of that equation in terms of λ_1, λ_2, and l_1, we ask for the probability of certain values of λ_1 and λ_2, we must let l_1 run through all possible values between $-\infty$ and $+\infty$ .3 The probability sought is then obtained by summing the terms corresponding to these l_1-values. Let it be denoted by $B(\lambda_1, \lambda_2)$, so that

$$
(24)\qquad B(\lambda_1, \lambda_2) = \int_{-\infty}^{+\infty} d l_1\, B(m_1, m_2, m_3, m_4).
$$

If we put

$$
B(m_1, m_2, m_3, m_4) = \frac{e^{-\frac{1}{2}\Phi}}{\sqrt{(2\pi)^3 s^3 p_1 p_2 p_3 p_4}},
$$

then

$$
(25)\qquad
\begin{aligned}
\Phi = {} & (a_1 + a_2 + a_3 + a_4)\, l_1^2 - 2\, l_1 \left[a_2 \lambda_1 + a_3 \lambda_2 + a_4 (\lambda_1 + \lambda_2)\right] \\
& + (a_2 + a_4)\, \lambda_1^2 + (a_3 + a_4)\, \lambda_2^2 + 2\, a_1 \lambda_1 \lambda_2,
\end{aligned}
$$

where

$$
(25^*)\qquad a_r = 1/s p_r \ (r = 1, 2, 3, 4).
$$

Applying the formula of integration

$$
(26)\qquad \int_{-\infty}^{+\infty} e^{-\frac{1}{2} h l_1^2}\, d l_1 = \sqrt{\frac{2\pi}{h}}
$$

we have

3. This, of course, is not strictly true, but may be used in practice as a good approximation. We must remember that l_1, l_2, l_3, l_4, and therefore likewise λ_1, λ_2, are considered as small numbers in relation to sp_1, sp_2, sp_3, sp_4; but that the squares of the l and λ are not so considered.

$$B(\lambda_1, \lambda_2) = \frac{e^{-\frac{1}{2}\Phi_1}}{\sqrt{(2\pi)^2 s^8 p_1 p_2 p_3 p_4 (a_1 + a_2 + a_3 + a_4)}}, \tag{27}$$

where

$$\begin{aligned} \Phi_1 = \{&\lambda_1^2 (a_2 + a_4)(a_1 + a_3) + \lambda_2^2 (a_3 + a_4)(a_1 + a_2) + \\ &+ 2\lambda_1 \lambda_2 (a_1 a_4 - a_2 a_3)\} : (a_1 + a_2 + a_3 + a_4), \end{aligned} \tag{28}$$

which may be written in the form

$$\Phi_1 = \frac{1}{1 - r^2} \left\{ \frac{\lambda_1^2}{\sigma_1^2} + \frac{\lambda_2^2}{\sigma_2^2} - \frac{2r\lambda_1\lambda_2}{\sigma_1 \sigma_2} \right\}. \tag{28*}$$

The quantities σ_1, σ_2, and r are then given by the following formulae:

$$\begin{aligned} \frac{1}{1-r^2} \frac{1}{\sigma_1^2} &= \frac{(a_2 + a_4)(a_1 + a_3)}{a_1 + a_2 + a_3 + a_4}, \\ \frac{1}{1-r^2} \frac{1}{\sigma_2^2} &= \frac{(a_3 + a_4)(a_1 + a_2)}{a_1 + a_2 + a_3 + a_4}, \\ \frac{r}{1-r^2} \frac{1}{\sigma_1 \sigma_2} &= -\frac{a_1 a_4 - a_2 a_3}{a_1 + a_2 + a_3 + a_4}. \end{aligned} \tag{29}$$

Therefore

$$\begin{aligned} r\frac{\sigma_1}{\sigma_2} &= -\frac{a_1 a_4 - a_2 a_3}{(a_2 + a_4)(a_1 + a_3)}, \\ r\frac{\sigma_2}{\sigma_1} &= -\frac{a_1 a_4 - a_2 a_3}{(a_3 + a_4)(a_1 + a_2)}, \end{aligned} \tag{29*}$$

and by multiplication

$$r^2 = \frac{(a_1 a_4 - a_2 a_3)^2}{(a_2 + a_4)(a_3 + a_4)(a_1 + a_3)(a_1 + a_2)}. \tag{29**}$$

The first two of these formulae shew that r has the same sign as the quantity $a_2a_3 - a_1a_4$. The coefficient of correlation may be positive or negative and is always numerically smaller than unity, as is seen immediately from (29**) or (29). Introducing the values of the a from (25*), the expression for r takes the form

$$r = \frac{p_1 p_4 - p_2 p_3}{\sqrt{(p_1+p_2)(p_1+p_3)(p_2+p_4)(p_3+p_4)}} \tag{30}$$

A rather lengthy algebraic manipulation yields the expression

$$1 - r^2 = \frac{s\,(a_1+a_2+a_3+a_4)\,a_1\,a_2\,a_3\,a_4}{(a_1+a_2)(a_1+a_3)(a_2+a_4)(a_3+a_4)}, \tag{31}$$

so that, by (29),

$$\sigma_1^2 = \frac{(a_1+a_2)(a_3+a_4)}{s\,a_1\,a_2\,a_3\,a_4},$$

$$\sigma_2^2 = \frac{(a_1+a_3)(a_2+a_4)}{s\,a_1\,a_2\,a_3\,a_4},$$

or, introducing p_1, p_2, p_3, p_4,

$$\begin{aligned} \sigma_1^2 &= s\,(p_1+p_2)(p_3+p_4), \\ \sigma_2^2 &= s\,(p_1+p_3)(p_2+p_4). \end{aligned} \tag{32}$$

These expressions shew that σ_1 gives the dispersion of the frequency curve obtained in answer to the problem: 'What is the probability of obtaining in s trials t_1 individuals with the attribute A?'

The quantity σ_2 has analogous signification. These frequency curves are obtained by integrating the expression (27) for $B(\lambda_1, \lambda_2)$ over all values of λ_2 (or λ_1) from $-\infty$ to $+\infty$.

I call the expression (30) for r the Bernoulli coefficient of correlation and designate it by r_B, since it corresponds to that correlation that exists between two attributes with Bernoullian dispersion.

The schema for investigating this correlation may conveniently be written in the following form (the notation was first introduced by Yule [23]).

Table 39.

Schema of correlation in homograde statistics.

The attributes A and B may occur simultaneously.

$(A\,B)$	$(a\,B)$	(B)
$(A\,b)$	$(a\,b)$	(b)
(A)	(a)	N

Here (AB) denotes the number of individuals simultaneously possessing the attributes A and B, (Ab) the number possessing the attribute A but not the attribute B, $(A) = (AB) + (Ab)$ the total number possessing the attribute A, etc.

The expression (30) for r_B can then be written in the following form:

$$r_B = \frac{(AB)(ab) - (Ab)(aB)}{\sqrt{(A)(B)(a)(b)}} \tag{33}$$

The BERNOULLI coefficient of correlation vanishes when

$$(AB):(Ab) = (aB):(ab),$$

i.e. when the ratio of the number of individuals with the attribute B to the number without this attribute is the same regardless of the presence of the attribute A. Obviously the attributes A and B have then nothing to do with one another.

Furthermore the coefficient r_B vanishes when

$$\begin{array}{lll} 1.\ (AB) = (Ab) = 0, & \text{so that} & (A) = 0, \\ 2.\ (AB) = (aB) = 0, & \text{"} \quad \text{"} & (B) = 0, \\ 3.\ (ab) = (Ab) = 0, & \text{"} \quad \text{"} & (b) = 0, \\ 4.\ (ab) = (aB) = 0, & \text{"} \quad \text{"} & (a) = 0. \end{array}$$

It is easy to see that in these cases no correlation can be proven to exist between the attributes A and B.

Contrariwise the BERNOULLI coefficient of correlation has the value +1 if

$$(Ab) = (aB) = 0,$$

so that all individuals possessing the attribute A have also the attribute B.

It has the value -1 if

$$(AB) = (ab) = 0,$$

so that all individuals possessing the attribute A lack the attribute B, and *vice versa*.

In all other cases r has a value greater than -1 and smaller than +1 .

If the value of λ_2 is known and the most probable value of λ_1 --which we shall call A_1--is sought, we have to differentiate $B(\lambda_1, \lambda_2)$ with respect to λ_2 and set the differential equal to zero. We then obtain

$$\Lambda_1 = r\frac{\sigma_1}{\sigma_2}\lambda_2. \tag{34}$$

If the value of λ_1 is given, the most probable corresponding value of λ_2 ($=\Lambda_2$) is obtained from the formula

$$\Lambda_2 = r\frac{\sigma_2}{\sigma_1}\lambda_1,$$

so that the regression coefficients are obtained as in heterograde statistics.

From (29*) are obtained the following expressions for the regression coefficients:

$$(35)\quad \begin{aligned} r\frac{\sigma_1}{\sigma_2} &= \frac{(AB)(ab)-(Ab)(aB)}{(B)(b)}, \\ r\frac{\sigma_1}{\sigma_2} &= \frac{(AB)(ab)-(Ab)(aB)}{(A)(a)}. \end{aligned}$$

97 As an application of these formulae I shall treat the following case:

At the general hospital in Copenhagen, at one time, the children afflicted with diphtheria were treated alternately with the serum of BEHRING and by older methods. Professor FIBIGER reports [8, p. 317] that in 239 cases treated with serum 8 children died, whereas out of 244 children treated without serum, 29 succumbed to the disease. How large is the BERNOULLI coefficient of correlation between serum and recovery?

Obviously we have here an example of homograde statistics, because a gradation of the attribute death cannot occur, and further, as far as was reported, the quantity of serum was the same in all the 239 cases.

Table 40.

Correlation between serum and recovery.

	Recovery	Death	
With serum	231	8	239
Without serum ...	215	29	244
	446	37	483

From formula (30) we have

$$r_B = \frac{231 \times 29 - 215 \times 8}{\sqrt{239 \times 446 \times 37 \times 244}} = +\,0.1605.$$

The coefficient of correlation between recovery and the serum of BEHRING is thus +0.160 . Perhaps a stronger correlation would have been expected from the figures. In this connexion it should be noted that even without serum a significant percentage of the cases of illness recovered. If it were a case of an illness from which, say, 100 per cent. deaths occurred without serum, but only p per cent. with serum, then obviously we would have

$$r_B = \sqrt{\frac{100 - p}{100 + p}}$$

and for small values of p

$$r_B = 1 - \frac{p}{100}.$$

With respect to the mean error I refer to the papers of YULE [24] and of WICKSELL [22].

YULE finds

$$s\,\varepsilon^2(r) = 1 - r^2 + \left(r + \frac{1}{2}r^3\right)\frac{[(A)-(a)]\,[(B)-(b)]}{\sqrt{(A)\,(a)\,(B)\,(b)}} - \\ - \frac{3}{4}r^2\left(\frac{[(A)-(a)]^2}{(A)\,(a)} + \frac{[(B)-(b)]^2}{(B)\,(b)}\right).$$

98 The attributes that we have treated up to now may occur simultaneously in one individual. Let us now consider the case where the attributes between which the correlation is sought are all homograde but cannot occur simultaneously in one individual. Such attributes are called alternative attributes.

If there are only two alternative attributes--A and B--the solution is simple. If we take N individuals and find that m_1 of these have the attribute A, we know then that $N - m_1$ individuals have the attribute B. The coefficient of correlation here has the value -1 . If, however, there are three attributes--A, B, and C--the solution is different. If, of N individuals, m_1 possess the attribute A, then of the remaining $N - m_1$ individuals a part have the attribute B and a part the attribute C. The problem will be to calculate the probable number of individuals with the attribute B or C. The same problem occurs with four or more attributes.

Let us take the case of four alternative attributes

Let s, the total number of individuals, be assumed constant.

Designating by M_1, M_2, M_3, M_4, the mean number of individuals with the attributes A, B, C, D, respectively, so that

$$M_1 + M_2 + M_3 + M_4 = s, \tag{36}$$

we define

$$p_1 = M_1 : s, \ p_2 = M_2 : s, \ p_3 = M_3 : s, \ p_4 = M_4 : s, \tag{36*}$$

thus considering p_1, p_2, p_3, p_4 as the respective probabilities of obtaining an individual of the group A, B, C, D, if an individual be chosen from the population at random. As an obvious consequence of (36),

$$p_1 + p_2 + p_3 + p_4 = 1. \tag{37}$$

If we now take an arbitrary sample of s individuals from the population and obtain m_1, m_2, m_3, m_4 individuals with the attributes A, B, C, D,

$$m_1 + m_2 + m_3 + m_4 = s.$$

If now, as in the previous problem, we set

$$m_r = s p_r + l_r \ \ (r = 1, 2, 3, 4),$$

so that from (37)

$$l_1 + l_2 + l_3 + l_4 = 0, \tag{38}$$

we may ask for the correlation between two of these numbers l, say between l_1 and l_2.

We obtain for the probability of the simultaneous deviations l_1, l_2, l_3, l_4, as in the previous problem,

$$B(m_1, m_2, m_3, m_4) = \frac{e^{-\frac{l_1^2}{2sp_1} - \frac{l_2^2}{2sp_2} - \frac{l_3^2}{2sp_3} - \frac{l_4^2}{2sp_4}}}{\sqrt{(2\pi)^3 s^3 p_1 p_2 p_3 p_4}} \tag{39}$$

Eliminating l_4 by means of (38), and integrating (39) from $-\infty$ to $+\infty$ with respect to l_3, we obtain the probability of simultaneous occurrence of the deviations l_1 and l_2.

This probability--$B(l_1, l_2)$--is

$$B(l_1, l_2) = \frac{e^{-\frac{1}{2}\Phi}}{\sqrt{(2\pi)^2 s^2 p_1 p_2 (p_3 + p_4)}},$$

where

$$\Phi = \frac{1}{1 - r_{12}^2}\left\{\frac{l_1^2}{\sigma_1^2} + \frac{l_2^2}{\sigma_2^2} - \frac{2 r_{12} l_1 l_2}{\sigma_1 \sigma_2}\right\}$$

and

$$\frac{1}{1 - r_{12}^2}\,\frac{1}{\sigma_1^2} = \frac{q_2}{s p_1 (p_3 + p_4)},$$

$$\frac{1}{1 - r_{12}^2}\,\frac{1}{\sigma_2^2} = \frac{q_1}{s p_2 (p_3 + p_4)}, \quad (q_i = 1 - p_i),$$

$$\frac{r_{12}}{1 - r_{12}^2}\,\frac{1}{\sigma_1 \sigma_2} = -\frac{1}{s(p_3 + p_4)},$$

whence follows

$$r_{12}\frac{\sigma_1}{\sigma_2} = -\frac{p_1}{q_2},$$

$$r_{12}\frac{\sigma_2}{\sigma_1} = -\frac{p_2}{q_1},$$

$$r_{12}^2 = \frac{p_1 p_2}{q_1 q_2}.$$

The first two of these equations shew that r is negative, so that

(40) $$r_{12} = -\sqrt{\frac{p_1 p_2}{q_1 q_2}},$$

an equation first given by PEARSON [17 , no. 92].

So we have

$$\sigma_1 = \sqrt{s p_1 q_1}, \quad \sigma_2 = \sqrt{s p_2 q_2},$$

$$\sigma_1 \sigma_2 r_{12} = -s p_1 p_2.$$

This treatment can obviously be extended to an arbitrary number of attributes in a population.

99 As a simple example of formula (40) we may take the following problem:

How large is the correlation between the number of spades and clubs at whist?

99 We here have two attributes that cannot occur simultaneously in one individual. We have

$$p_1 = p_2 = 1/4,$$

$$q_1 = q_2 = 3/4,$$

so that by formula (40)

$$r_{12} = -1/3.$$

To illuminate the concept of correlation, the significance of this result should be more completely explained.

The total number of cards for each player is 13 . The mean number of cards of each suit is 3.25 . If a player has more than 4 spades, then the average number of cards of the remaining suits must be less than 3 . If, say, he receives 7 spades, then on the average he must have 2 clubs, 2 diamonds, and 2 hearts. To an excess of 3.75 spades (over the theoretic mean of 3.25 spades) corresponds an average deficiency of 1.25 cards in each of the other three suits. The excess in one suit is uniformly divided among the other suits. This connexion between the number of cards in different suits is technically expressed by the statement that the coefficient of correlation between the number of cards in two different suits at whist is negative and has the value -1/3 .

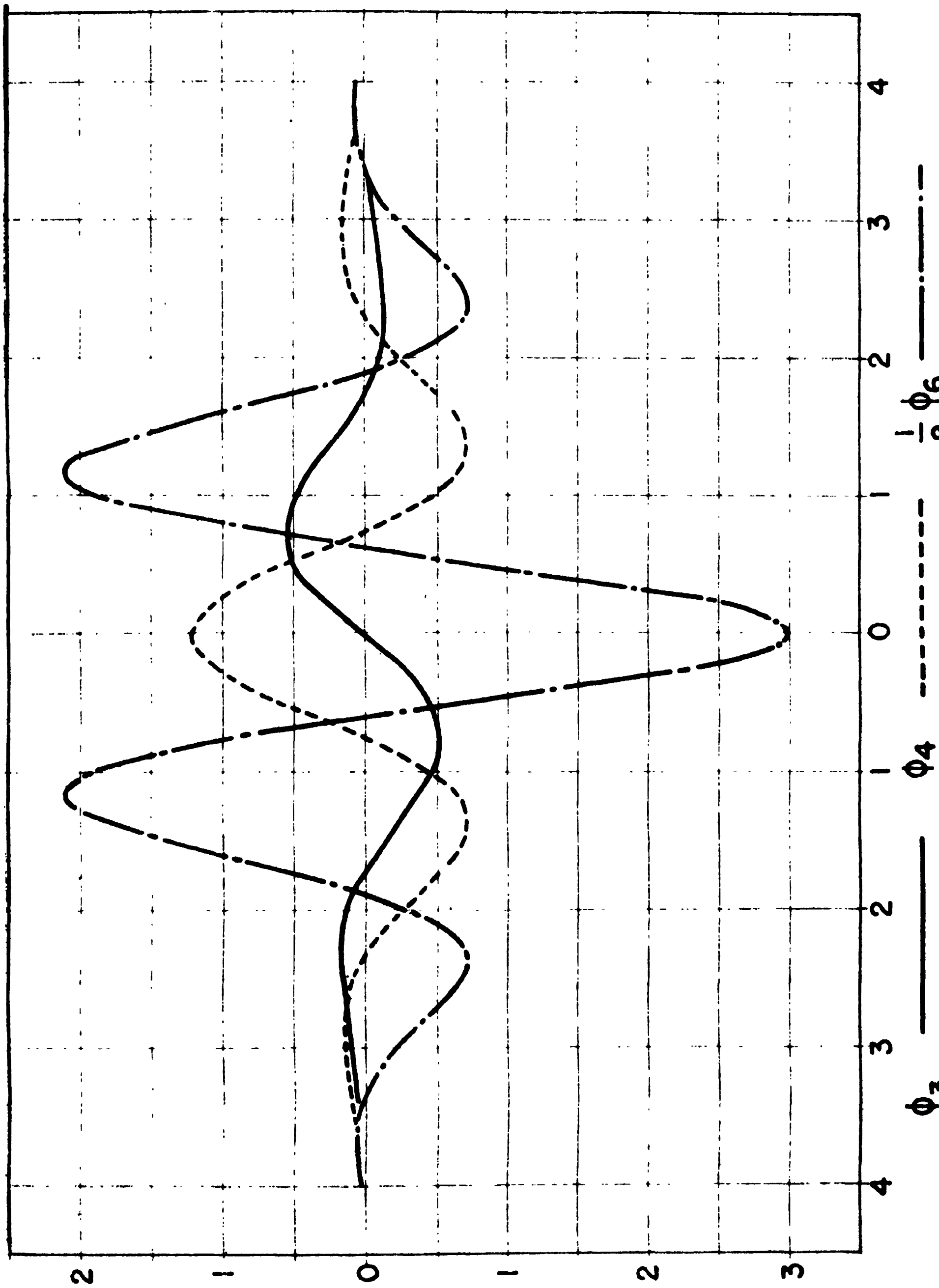

ϕ_3
ϕ_4
$\frac{1}{2}\phi_6$

Table 41.

u	$-x=-R(u)$		u	$-x=-R(u)$	
.01	2.3263	.99	.26	0.6433	.74
.02	2.0537	.98	.27	0.6128	.73
.03	1.8808	.97	.28	0.5828	.72
.04	1.7507	.96	.29	0.5534	.71
.05	1.6449	.95	.30	0.5244	.70
.06	1.5548	.94	.31	0.4959	.69
.07	1.4758	.93	.32	0.4677	.68
.08	1.4051	.92	.33	0.4399	.67
.09	1.3408	.91	.34	0.4125	.66
.10	1.2816	.90	.35	0.3853	.65
.11	1.2265	.89	.36	0.3585	.64
.12	1.1750	.88	.37	0.3319	.63
.13	1.1264	.87	.38	0.3055	.62
.14	1.0803	.86	.39	0.2793	.61
.15	1.0364	.85	.40	0.2533	.60
.16	0.9945	.84	.41	0.2275	.59
.17	0.9542	.83	.42	0.2019	.58
.18	0.9154	.82	.43	0.1764	.57
.19	0.8779	.81	.44	0.1510	.56
.20	0.8416	.80	.45	0.1257	.55
.21	0.8064	.79	.46	0.1004	.54
.22	0.7722	.78	.47	0.0753	.53
.23	0.7388	.77	.48	0.0502	.52
.24	0.7063	.76	.49	0.0251	.51
.25	0.6745	.75	.50	0.0000	.50
	$x=R(u)$	u		$x=R(u)$	u

Table 42.

X	ϕ_{-1}	ϕ_0	$-\phi_2$	ϕ_3	ϕ_4	$-\phi_5$	$-\phi_6$
0.00	.5000000	.3989423	.3989423	.0000000	1.1968	.0000000	5.9841
.01	.5039894	.3989223	.3988824	.0119673	1.1965	.0598344	5.9820
.02	.5079783	.3988625	.3987030	.0239286	1.1956	.1196268	5.9758
.03	.5119665	.3987628	.3984027	.0358779	1.1941	.1793356	5.9653
.04	.5159534	.3986233	.3979855	.0478093	1.1920	.2389189	5.9507
.05	.5199388	.3984439	.3974478	.0597168	1.1894	.2983350	5.9319
.06	.5239222	.3982248	.3967912	.0715945	1.1861	.3575425	5.9089
.07	.5279032	.3979661	.3960160	.0834364	1.1822	.4165000	5.8819
.08	.5318814	.3976677	.3951226	.0952366	1.1777	.4751665	5.8507
.09	.5358564	.3973298	.3941115	.1069894	1.1727	.5335011	5.8155
0.10	.5398278	.3969525	.3929830	.1186888	1.1671	.5914633	5.7763
.11	.5437953	.3960802	.3917379	.1303291	1.1609	.6490128	5.7330
.12	.5477584	.3969802	.3903767	.1419044	1.1541	.7061100	5.6858
.13	.5517168	.3955854	.3889000	.1534092	1.1468	.7627152	5.6346
.14	.5556700	.3950517	.3873087	.1648377	1.1388	.8187897	5.5796
.15	.5596177	.3944793	.3856035	.1761843	1.1304	.8742948	5.5208
.16	.5635595	.3938684	.3837853	.1874435	1.1214	.9291925	5.4581
.17	.5674949	.3932190	.3818550	.1986098	1.1118	.9834455	5.3918
.18	.5714237	.3925315	.3798135	.2096778	1.1017	1.0370167	5.3218
.19	.5753454	.3918060	.3776618	.2206420	1.0911	1.0898701	5.2482
0.20	.5792597	.3910427	.3754010	.2314972	1.0799	1.1419698	5.1711
.21	.5831662	.3902419	.3730322	.2422384	1.0682	1.1932810	5.0905
.22	.5870644	.3894038	.3705566	.2528601	1.0560	1.2437694	5.0066
.23	.5909541	.3885286	.3679754	.2633575	1.0434	1.2934014	4.9193
.24	.5948349	.3876166	.3652899	.2737256	1.0302	1.3421443	4.8288
.25	.5987063	.3866681	.3625014	.2839594	1.0165	1.3899662	4.7351
.26	.6025681	.3856834	.3596112	.2940543	1.0024	1.4368357	4.6383
.27	.6064199	.3846627	.3566208	.3040054	.9878	1.4827226	4.5386
.28	.6102612	.3836063	.3535316	.3138084	.9727	1.5275974	4.4359
.29	.6140919	.3825146	.3503451	.3234585	.9572	1.5714315	4.3304
0.30	.6179114	.3813878	.3470629	.3329516	.9413	1.6141972	4.2223
.31	.6217195	.3802264	.3436868	.3422832	.9250	1.6558679	4.1114
.32	.6255158	.3790305	.3402178	.3514492	.9082	1.6964176	3.9981
.33	.0293000	.3778007	.3366582	.3604456	.8910	1.7358217	3.8823
.34	.6330717	.3765372	.3330095	.3692685	.8735	1.7740562	3.7642
.35	.6368307	.3754203	.3292734	.3779139	.8556	1.8110984	3.6439
.36	.6405764	.3739106	.3254518	.3863783	.8373	1.8469264	3.5214
.37	.6443088	.3752403	.3215465	.3946579	.8186	1.8815197	3.3969
.38	.6480273	.3711539	.3175593	.4027495	.7996	1.9148584	3.2705
.39	.6517317	.3697277	.3134921	.4106495	.7803	1.9469240	3.1423
0.40	.6554217	.3682701	.3093469	.4183549	.7607	1.9776990	3.0124
.41	.6590970	.3667817	.3051257	.4258625	.7408	2.0071670	2.8809
.42	.6627573	.3652627	.3008303	.4331694	.7206	2.0353127	2.7480
.43	.6664022	.3637136	.2964630	.4402728	.7001	2.0621218	2.6136
.44	.6700314	.3621349	.2920256	.4471699	.6793	2.0875814	2.4781
.45	.6736448	.3605270	.2875203	.4538584	.6583	2.1116795	2.3414
.46	.6772419	.3588903	.2829491	.4603357	.6371	2.1344054	2.3036
.47	.6808225	.3572253	.2783143	.4665995	.6156	2.1557493	2.0650
.48	.6843863	.3555325	.2736178	.4726478	.5940	2.1757028	1.9256
.49	.6879331	.3538124	.2688620	.4784785	.5721	2.1942585	1.7855
0.50	.6914625	.3520653	.2640490	.4840898	.5501	2.2114103	1.6448
$-X$	$1-\phi_{-1}$	ϕ_0	$-\phi_2$	$-\phi_3$	ϕ_4	ϕ_5	$-\phi_6$

X	ϕ_{-1}	ϕ_0	$-\phi_2$	ϕ_3	ϕ_4	$-\phi_5$	ϕ_6
0.50	.6914625	.3520653	.2640490	.4840498	+.5501	2.2114103	-1.6448
.51	.6949743	.3502919	.2591810	.4894800	.5279	2.2271531	1.5037
.52	.6984682	.3484925	.2542601	.4946475	.5056	2.2414831	1.3622
.53	.7019440	.3466677	.2492888	.4995908	.4831	2.2543974	1.2206
.54	.7054015	.3448180	.2442691	.5043087	.4605	2.2658944	1.0788
.55	.7088403	.3429439	.2392033	.5088001	.4378	2.2759738	0.9371
.56	.7122603	.3410458	.2340938	.5130638	.4150	2.2846361	0.7954
.57	.7156612	.3391243	.2289428	.5170991	.3921	2.2918832	0.6540
.58	.7190427	.3371799	.2237526	.5209052	.3691	2.2977214	0.5130
.59	.7224047	.3352132	.2185255	.5244816	.3461	2.3021445	0.3724
0.60	.7257469	.3332246	.2132637	.5278278	+.3231	2.3051678	-0.2324
.61	.7290691	.3312147	.2079697	.5309434	.3000	2.3067942	-0.0930
.62	.7323711	.3291840	.2026456	.5338284	.2770	2.3070309	+0.0455
.63	.7356527	.3271330	.1972939	.5364827	.2539	2.3058863	0.1832
.64	.7389137	.3250623	.1919168	.5389064	.2309	2.3033698	0.3199
.65	.7421539	.3229724	.1865165	.5410998	.2078	2.2994918	0.4555
.66	.7453731	.3208638	.1810955	.5430633	.1849	2.2942637	0.5899
.67	.7485711	.3187371	.1756560	.5447973	.1620	2.2876984	0.7230
.68	.7517478	.3165929	.1702003	.5463026	.1391	2.2798088	0.8547
.69	.7549029	.3144317	.1647307	.5475799	.1164	2.2706095	0.9849
0.70	.7580363	.3122539	.1592495	.5486302	+.0937	2.2601158	+1.1135
.71	.7611479	.3100603	.1537589	.5494544	.0712	2.2483442	1.2405
.72	.7642375	.3078513	.1482612	.5500539	.0487	2.2353116	1.3657
.73	.7673049	.3056274	.1427586	.5504298	.0265	2.2210363	1.4890
.74	.7703500	.3033893	.1372533	.5505836	+.0043	2.2055371	1.6105
.75	.7733726	.3011374	.1317476	.5505169	-.0176	2.1888339	1.7299
.76	.7763727	.2988724	.1262437	.5502313	.0394	2.1709472	1.8471
.77	.7793501	.2965948	.1207437	.5497286	.0611	2.1518983	1.9623
.78	.7823046	.2943050	.1152498	.5490107	.0825	2.1317094	2.0751
.79	.7852361	.2920038	.1097642	.5480797	.1037	2.1104035	2.1857
0.80	.7881446	.2896916	.1042889	.5469377	-.1247	2.0880040	+2.2938
.81	.7910299	.2873689	.0988262	.5455868	.1454	2.0645353	2.3995
.82	.7938919	.2850364	.0933779	.5440295	.1660	2.0400223	2.5027
.83	.7967306	.2826945	.0879463	.5422682	.1862	2.0144905	2.6032
.84	.7995458	.2803438	.0825332	.5403055	.2063	1.9879662	2.7012
.85	.8023375	.2779849	.0771408	.5381440	.2260	1.9604760	2.7964
.86	.8051055	.2756182	.0717710	.5357864	.2455	1.9320473	2.8889
.87	.8078498	.2732444	.0664257	.5332357	.2646	1.9027078	2.9785
.88	.8105703	.2708640	.0611069	.5304947	.2835	1.8724859	3.0654
.89	.8132671	.2684774	.0558165	.5275664	.3021	1.8414102	3.1493
0.90	.8159399	.2660852	.0505562	.5244540	-.3203	1.8095101	+3.2303
.91	.8185887	.2636880	.0453280	.5211607	.3383	1.7768150	3.3083
.92	.8212136	.2612863	.0401336	.5176897	.3559	1.7433549	3.3833
.93	.8238145	.2588805	.0349748	.5140443	.3731	1.7091600	3.4552
.94	.8263912	.2564713	.0298533	.5102281	.3901	1.6742610	3.5241
.95	.8289439	.2540591	.0247708	.5062444	.4066	1.6386888	3.5899
.96	.8314724	.2516443	.0197289	.5020969	.4228	1.6024744	3.6525
.97	.8339768	.2492277	.0147294	.4977891	.4387	1.5656491	3.7120
.98	.8364569	.2468095	.0097737	.4933248	.4541	1.5282446	3.7684
.99	.8389129	.2443904	.0048634	.4887076	.4692	1.4902924	3.8215
1.00	.8413447	.2419707	.0000000	.4839414	-.4839	1.4518243	+3.8715
$-X$	$1-\phi_{-1}$	ϕ_0	$-\phi_2$	$-\phi_3$	ϕ_4	ϕ_5	ϕ_6

X	ϕ_{-1}	ϕ_0	ϕ_2	ϕ_3	$-\phi_4$	ϕ_5	ϕ_6
1.00	.8413447	.2419707	.0000000	.4839414	.4839	−1.4518243	3.8715
.01	.8437524	.2395511	.0048150	.4790301	.4983	1.4128724	3.9183
.02	.8461358	.2371320	.0095801	.4739774	.5122	1.3734685	3.9619
.03	.8484950	.2347138	.0142941	.4687874	.5257	1.3336446	4.0023
.04	.8508300	.2322970	.0189554	.4634641	.5389	1.2934327	4.0395
.05	.8531409	.2298821	.0235629	.4580114	.5516	1.2528649	4.0735
.06	.8554277	.2274696	.0281152	.4524335	.5639	1.2119731	4.1043
.07	.8576903	.2250599	.0326112	.4467343	.5758	1.1707892	4.1319
.08	.8599289	.2226535	.0370495	.4409181	.5873	1.1293449	4.1564
.09	.8621434	.2202508	.0414292	.4349889	.5984	1.0876718	4.1777
1.10	.8643339	.2178522	.0457490	.4289509	.6091	−1.0458016	4.1958
.11	.8665005	.2154582	.0500078	.4228084	.6193	1.0037653	4.2109
.12	.8686431	.2130691	.0542048	.4165655	.6292	0.9615942	4.2228
.13	.8707619	.2106856	.0583388	.4102265	.6386	0.9193191	4.2317
.14	.8728568	.2083078	.0624090	.4037955	.6476	0.8769705	4.2375
.15	.8749281	.2059363	.0664144	.3972768	.6561	0.8345788	4.2403
.16	.8769756	.2035714	.0703543	.3906747	.6642	0.7921740	4.2401
.17	.8789995	.2012135	.0742277	.3839933	.6720	0.7497857	4.2370
.18	.8809999	.1988631	.0780339	.3772370	.6792	0.7074432	4.2310
.19	.8829768	.1965205	.0817722	.3704099	.6861	0.6651754	4.2221
1.20	.8849303	.1941861	.0854419	.3635163	.6925	−0.6230110	4.2103
.21	.8868606	.1918602	.0890423	.3565604	.6986	0.5809780	4.1958
.22	.8887676	.1895432	.0925729	.3495464	.7042	0.5391040	4.1785
.23	.8906514	.1872354	.0960330	.3424785	.7093	0.4974163	4.1586
.24	.8925123	.1849373	.0994223	.3353608	.7141	0.4559416	4.1359
.25	.8943502	.1826491	.1027401	.3281976	.7185	0.4147062	4.1107
.26	.8961653	.1803712	.1059861	.3209929	.7224	0.3737357	4.0830
.27	.8979577	.1781038	.1091598	.3137508	.7259	0.3330554	4.0527
.28	.8997274	.1758474	.1122610	.3064753	.7292	0.2926899	4.0200
.29	.9014747	.1736022	.1152893	.2991707	.7318	0.2526633	3.9849
1.30	.9031995	.1713686	.1182443	.2918407	.7341	−0.2129992	3.9475
.31	.9049021	.1691468	.1211260	.2844895	.7361	0.1737203	3.9079
.32	.9065825	.1669370	.1239341	.2771208	.7376	0.1348491	3.8660
.33	.9082409	.1647397	.1266684	.2697387	.7388	0.0964073	3.8220
.34	.9098773	.1625551	.1293288	.2623470	.7395	0.0584158	3.7759
.35	.9114920	.1603833	.1319153	.2549493	.7399	−0.0208953	3.7278
.36	.9130850	.1582248	.1344278	.2475496	.7400	+0.0161346	3.6778
.37	.9146565	.1560797	.1368663	.2401516	.7396	0.0526547	3.6259
.38	.9162067	.1539483	.1392308	.2327587	.7389	0.0886465	3.5722
.39	.9177356	.1518308	.1415215	.2253748	.7378	0.1240922	3.5167
1.40	.9192433	.1497275	.1437384	.2180032	.7364	+0.1589746	3.4595
.41	.9207302	.1476385	.1458816	.2106475	.7347	0.1932775	3.4008
.42	.9221962	.1455641	.1479514	.2033112	.7326	0.2269849	3.3405
.43	.9236415	.1435046	.1499479	.1959975	.7301	0.2600818	3.2787
.44	.9250663	.1414600	.1518714	.1887099	.7274	0.2925538	3.2155
.45	.9264707	.1394306	.1537222	.1814515	.7243	0.3243874	3.1510
.46	.9278550	.1374165	.1555006	.1742255	.7209	0.3555695	3.0852
.47	.9292191	.1354181	.1572068	.1670351	.7172	0.3860879	3.0183
.48	.9305634	.1334353	.1588414	.1598832	.7132	0.4159310	2.9502
.49	.9318879	.1314684	.1604046	.1527730	.7088	0.4450880	2.8810
1.50	.9331928	.1295176	.1618970	.1457073	.7043	+0.4735487	2.8109
$-X$	$1-\phi_{-1}$	ϕ_0	ϕ_2	$-\phi_3$	$-\phi_4$	$-\phi_5$	ϕ_6

X	ϕ_{-1}	ϕ_0	ϕ_2	ϕ_3	$-\phi_4$	ϕ_5	ϕ_6
1.50	.9331928	.1295176	.1618970	+.1457073	.7043	.4735487	+2.1809
.51	.9344783	.1275830	.1633189	.1386889	.6994	.5013037	2.7399
.52	.9357445	.1256646	.1646709	.1317207	.6942	.5283442	2.6881
.53	.9369916	.1237628	.1659535	.1248052	.6888	.5546623	2.5954
.54	.9382198	.1218775	.1671672	.1179453	.6831	.5802505	2.5221
.55	.9394292	.1200090	.1683126	.1111433	.6772	.6051022	2.4481
.56	.9406201	.1181573	.1693903	.1044019	.6710	.6292115	2.3736
.57	.9417924	.1163225	.1704009	.0977234	.6646	.6525730	2.2986
.58	.9429466	.1145048	.1713450	.0911101	.6580	.6751821	2.2232
.59	.9440826	.1127042	.1722233	.0845643	.6511	.6970349	2.1474
1.60	.9452007	.1109208	.1730365	+.0780883	.6441	.7181281	+2.0712
.61	.9463011	.1091548	.1774704	.0716840	.6368	.7384591	1.9949
.62	.9473839	.1074061	.1744704	.0653536	.6293	.7580259	1.9184
.63	.9484493	.1056748	.1750926	.0590990	.6216	.7768271	1.8418
.64	.9494974	.1039611	.1756527	.0529220	.6137	.7948621	1.7652
.65	.9505285	.1022649	.1761513	.0468246	.6057	.8121307	1.6886
.66	.9515428	.1005864	.1765894	.0408083	.5975	.8286335	1.6120
.67	.9525403	.0989255	.1769678	.0348749	.5891	.8443716	1.5356
.68	.9535213	.0972823	.1772872	.0290259	.5806	.8593466	1.4594
.69	.9544860	.0956568	.1775486	.0232629	.5720	.8735609	1.3835
1.70	.9554345	.0940491	.1777528	+.0175872	.5632	.8870173	+1.3078
.71	.9563671	.0924591	.1779006	.0120002	.5542	.8997192	1.2326
.72	.9572838	.0908870	.1779931	.0065031	.5452	.9116705	1.1577
.73	.9581849	.0893326	.1780310	+.0010973	.5360	.9228758	1.0834
.74	.9590705	.0877961	.1780153	−.0042163	.5267	.9333399	1.0095
.75	.9599408	.0862773	.1779470	.0094366	.5173	.9430684	0.9363
.76	.9607961	.0847764	.1778269	.0145625	.5079	.9520672	0.8636
.77	.9616364	.0832932	.1776560	.0195933	.4983	.9603429	0.7916
.78	.9624620	.0818278	.1774353	.0245280	.4886	.9679023	0.7204
.79	.9632730	.0803801	.1771658	.0293660	.4789	.9747527	0.6499
1.80	.9640697	.0789502	.1768484	−.0341065	.4692	.9809005	+0.5801
.81	.9648521	.0775379	.1764840	.0387489	.4593	.9863584	0.5113
.82	.9656205	.0761433	.1760737	.0432926	.4494	.9911304	0.4433
.83	.9663750	.0747663	.1756185	.0477373	.4395	.9952272	0.3762
.84	.9671159	.0734068	.1751193	.0520824	.4295	.9986579	0.3101
.85	.9678432	.0720649	.1745772	.0563277	.4195	1.0014325	0.2450
.86	.9685572	.0707404	.1739931	.0604728	.4095	1.0035609	0.1809
.87	.9692581	.0694333	.1733680	.0645176	.3995	1.0050535	0.1178
.88	.9699460	.0681436	.1727031	.0684619	.3894	1.0059211	+0.0559
.89	.9706210	.0668711	.1719991	.0723058	.3793	1.0061747	−0.0050
1.90	.9712834	.0656158	.1712573	−.0760487	.3693	1.0058255	−0.0647
.91	.9719334	.0643777	.1704785	.0796912	.3592	1.0048851	0.1232
.92	.9725711	.0631566	.1696638	.0832333	.3492	1.0033654	0.1805
.93	.9731966	.0619524	.1688142	.0866750	.3392	1.0012783	0.2367
.94	.9738102	.0607652	.1679306	.0900165	.3292	.9986361	0.2916
.95	.9744119	.0595947	.1670142	.0932583	.3192	.9954514	0.3452
.96	.9750021	.0584409	.1660658	.0964004	.3093	.9917367	0.3975
.97	.9755808	.0573038	.1650865	.0994434	.2994	.9875048	0.4486
.98	.9761482	.0561831	.1640772	.1023877	.2895	.9827689	0.4984
.99	.9767045	.0550789	.1630391	.1052337	.2797	.9775420	0.5468
2.00	.9772499	.0539910	.1619729	−.1079819	.2700	.9718374	−0.5939
$-X$	$1-\phi_{-1}$	ϕ_0	ϕ_2	$-\phi_3$	$-\phi_4$	$-\phi_5$	ϕ_6

X	$\phi_{.1}$	ϕ_0	ϕ_2	$-\phi_3$	ϕ_4	ϕ_5	$-\phi_6$
2.00	.9772499	.0539910	.1619729	.1079819	−.2700	.9718374	.5939
.01	.9777844	.0529194	.1608797	.1106330	.2603	.9656685	.6397
.02	.9783083	.0518636	.1597606	.1131875	.2506	.9590487	.6841
.03	.9788217	.0508239	.1586163	.1156461	.2411	.9519917	.7271
.04	.9793248	.0498001	.1574480	.1180095	.2316	.9445112	.7688
.05	.9798178	.0487920	.1562564	.1202784	.2222	.9366207	.8091
.06	.9803007	.0477996	.1550427	.1224537	.2129	.9283342	.8480
.07	.9807738	.0468226	.1538077	.1245362	.2036	.9196653	.8855
.08	.9812372	.0458611	.1525523	.1265267	.1945	.9106279	.9217
.09	.9816911	.0449148	.1512774	.1284261	.1854	.9012359	.9565
2.10	.9821356	.0439836	.1499841	.1302354	−.1765	.8915031	.9899
.11	.9825708	.0430674	.1486730	.1319556	.1676	.8814432	1.0219
.12	.9829970	.0421661	.1473452	.1335876	.1588	.8710700	1.0525
.13	.9834142	.0412795	.1460016	.1351325	.1502	.8603974	1.0818
.14	.9838226	.0404076	.1446429	.1365914	.1416	.8494389	1.1097
.15	.9842224	.0395500	.1432700	.1379654	.1332	.8382082	1.1362
.16	.9846137	.0387069	.1418839	.1392555	.1249	.8267189	1.1614
.17	.9849966	.0378779	.1404852	.1404630	.1167	.8149844	1.1853
.18	.9853713	.0370629	.1390749	.1415889	.1086	.8030181	1.2078
.19	.9857379	.0362619	.1376537	.1426346	.1006	.7908333	1.2290
2.20	.9860966	.0354746	.1362224	.1436012	−.0927	.7784431	1.2489
.21	.9864474	.0347009	.1347819	.1444899	.0840	.7658606	1.2674
.22	.9867906	.0339408	.1333329	.1453020	.0774	.7530987	1.2847
.23	.9871263	.0331939	.1318761	.1460389	.0700	.7401701	1.3008
.24	.9874545	.0324603	.1304124	.1467017	.0626	.7270874	1.3156
.25	.9877755	.0317397	.1289423	.1472918	.0554	.7138632	1.3291
.26	.9880894	.0310319	.1274668	.1478106	.0483	.7005097	1.3414
.27	.9883962	.0303370	.1259864	.1482592	.0414	.6870390	1.3525
.28	.9886962	.0296546	.1245018	.1486392	.0346	.6734631	1.3625
.29	.9889893	.0289847	.1230138	.1489519	.0279	.6597938	1.3712
2.30	.9892759	.0283270	.1215230	.1491985	−.0214	.6460426	1.3788
.31	.9895559	.0276816	.1200300	.1493806	.0150	.6322208	1.3853
.32	.9898296	.0270481	.1185356	.1494994	.0088	.6183398	1.3907
.33	.9900969	.0264265	.1170403	.1495564	−.0027	.6044103	1.3950
.34	.9903581	.0258166	.1155447	.1495529	+.0033	.5904432	1.3982
.35	.9906133	.0252182	.1140494	.1494905	.0092	.5764490	1.4004
.36	.9908625	.0246313	.1125550	.1493703	.0148	.5624381	1.4016
.37	.9911060	.0240556	.1110622	.1491939	.0204	.5484204	1.4018
.38	.9913437	.0234910	.1095714	.1489627	.0258	.5344059	1.4010
.39	.9915758	.0229374	.1080831	.1486781	.0311	.5204041	1.3992
2.40	.9918025	.0223945	.1065980	.1483414	+.0362	.5064245	1.3965
.41	.9920237	.0218624	.1051164	.1479540	.0412	.4924762	1.3930
.42	.9922397	.0213407	.1036390	.1475174	.0461	.4785681	1.3885
.43	.9924506	.0208294	.1021663	.1470330	.0508	.4647089	1.3832
.44	.9926564	.0203284	.1006985	.1465021	.0554	.4509069	1.3771
.45	.9928572	.0198374	.0992364	.1459261	.0598	.4371704	1.3701
.46	.9930531	.0193563	.0977802	.1453063	.0641	.4235072	1.3624
.47	.9932443	.0188850	.0963304	.1446443	.0683	.4099250	1.3539
.48	.9934309	.0184233	.0948874	.1439412	.0723	.3964313	1.3447
.49	.9936128	.0179711	.0934517	.1431985	.0762	.3830332	1.3348
2.50	.9937903	.0175283	.0920236	.1424174	+.0800	.3697376	1.3242
$-X$.	$1-\phi_{.1}$	ϕ_0	ϕ_2	ϕ_3	ϕ_4	$-\phi_5$	$-\phi_6$

X	ϕ_{-1}	ϕ_0	ϕ_2	$-\phi_3$	ϕ_4	ϕ_5	$-\phi_6$
2.50	.9937903	.0175283	.0920236	.1424174	.0800	+.3697376	1.3242
.51	.9939634	.0170947	.0906035	.1415994	.0836	.3565512	1.3130
.52	.9941323	.0166701	.0891917	.1407458	.0871	.3434804	1.3011
.53	.9942969	.0162545	.0877887	.1398578	.0905	.3305314	1.2886
.54	.9944574	.0158476	.0863947	.1389367	.0937	.3177100	1.2756
.55	.9946139	.0154493	.0850100	.1379839	.0968	.3050220	1.2620
.56	.9947664	.0150596	.0836351	.1370006	.0998	.2924278	1.2478
.57	.9949151	.0146782	.0822701	.1359880	.1027	.2800675	1.2332
.58	.9950600	.0143051	.0809154	.1349474	.1054	.2678110	1.2180
.59	.9952012	.0139401	.0795713	.1338800	.1080	.2557081	1.2025
2.60	.9953388	.0135830	.0782379	.1327871	.1105	+.2437632	1.1864
.61	.9954729	.0132337	.0769156	.1316698	.1129	.2319805	1.1700
.62	.9956035	.0128921	.0756046	.1305293	.1152	.2203640	1.1532
.63	.9957308	.0125581	.0743051	.1293667	.1173	.2089174	1.1360
.64	.9958547	.0122315	.0730173	.1281833	.1194	.1976441	1.1185
.65	.9959754	.0119122	.0717415	.1269800	.1213	.1865476	1.1007
.66	.9960930	.0116001	.0704778	.1257582	.1231	.1756308	1.0826
.67	.9962074	.0112951	.0692264	.1245187	.1248	.1648966	1.0642
.68	.9963189	.0109969	.0679875	.1232628	.1264	.1543476	1.0456
.69	.9964274	.0107056	.0667612	.1219915	.1279	.1439862	1.0267
2.70	.9965330	.0104209	.0655477	.1207057	.1293	+.1338145	1.0076
.71	.9966358	.0101428	.0643471	.1194066	.1306	.1238346	.9883
.72	.9967359	.0098712	.0631596	.1180950	.1317	.1140482	.9689
.73	.9968333	.0096058	.0619852	.1167721	.1328	.1044569	.9493
.74	.9969280	.0093466	.0608242	.1154387	.1338	.0950621	.9296
.75	.9970202	.0090936	.0596765	.1140958	.1347	.0858649	.9098
.76	.9971099	.0088465	.0585423	.1127443	.1355	.0768664	.8899
.77	.9971972	.0086052	.0574216	.1113851	.1363	.0680674	.8699
.78	.9972821	.0083697	.0563146	.1100192	.1369	.0594685	.8499
.79	.9973646	.0081398	.0552213	.1086472	.1375	.0510701	.8298
2.80	.9974449	.0079155	.0541417	.1072702	.1379	+.0428726	.8097
.81	.9975229	.0076965	.0530759	.1058889	.1383	.0348761	.7896
.82	.9975988	.0074829	.0520239	.1045041	.1386	.0270805	.7695
.83	.9976726	.0072744	.0509858	.1031165	.1387	.0194857	.7495
.84	.9977443	.0070711	.0499616	.1017271	.1390	.0120913	.7294
.85	.9978140	.0068728	.0489513	.1003364	.1391	+.0048967	.7095
.86	.9978818	.0066793	.0479549	.0989452	.1391	−.0020973	.6896
.87	.9979476	.0064907	.0469724	.0975542	.1391	.0088954	.6698
.88	.9980116	.0063067	.0460038	.0961642	.1389	.0154947	.6501
.89	.9980738	.0061274	.0450491	.0947756	.1388	.0218974	.6305
2.90	.9981342	.0059525	.0441083	.0933893	.1385	−.0281048	.6110
.91	.9981929	.0057821	.0431813	.0920057	.1382	.0341182	.5917
.92	.9982498	.0056160	.0422681	.0906256	.1378	.0399389	.5725
.93	.9983052	.0054541	.0413688	.0892495	.1374	.0455685	.5534
.94	.9983589	.0052963	.0404831	.0878779	.1369	.0510086	.5346
.95	.9984111	.0051426	.0396112	.0865114	.1364	.0562610	.5159
.96	.9984618	.0049929	.0387529	.0851506	.1358	.0613274	.4974
.97	.9985110	.0048470	.0379082	.0837959	.1351	.0662098	.4791
.98	.9985588	.0047050	.0370769	.0824478	.1345	.0709101	.4610
.99	.9986051	.0045666	.0362592	.0811067	.1337	.0754306	.4431
3.00	.9986501	.0044318	.0354548	.0797733	.1330	−.0797733	.4255
$-X$	$1-\phi_{-1}$	ϕ_0	ϕ_2	ϕ_3	ϕ_4	$-\phi_5$	$-\phi_6$

X	ϕ_{-1}	ϕ_0	ϕ_2	$-\phi_3$	ϕ_4	$-\phi_5$	ϕ_6
3.00	.9986501	.0044318	.0354548	.0797733	.1330	.0797733	−.4255
.01	.9986938	.0043007	.0346637	.0784478	.1321	.0839405	.4080
.02	.9987361	.0041729	.0338858	.0771307	.1313	.0879345	.3908
.03	.9987772	.0040486	.0331210	.0758224	.1304	.0917578	.3739
.04	.9988171	.0039276	.0323693	.0745232	.1294	.0954127	.3572
.05	.9988558	.0038098	.0316305	.0732336	.1285	.0989018	.3407
.06	.9988933	.0036951	.0309046	.0719539	.1275	.1022276	.3245
.07	.9989297	.0035836	.0301914	.0706844	.1264	.1053928	.3086
.08	.9989650	.0034751	.0294909	.0694255	.1254	.1084001	.2929
.09	.9989992	.0033695	.0288029	.0681774	.1243	.1112521	.2775
3.10	.9990324	.0032668	.0281273	.0669404	.1231	.1139516	−.2624
.11	.9990646	.0031669	.0274640	.0657148	.1220	.1165014	.2476
.12	.9990957	.0030698	.0268130	.0645009	.1208	.1189042	.2330
.13	.9991260	.0029754	.0261740	.0632988	.1196	.1211631	.2188
.14	.9991553	.0028835	.0255470	.0621089	.1184	.1232807	.2048
.15	.9991836	.0027943	.0249318	.0609312	.1171	.1252601	.1911
.16	.9992112	.0027075	.0243283	.0597662	.1159	.1271041	.1777
.17	.9992378	.0026231	.0237364	.0586138	.1146	.1288157	.1646
.18	.9992636	.0025412	.0231560	.0574743	.1133	.1303978	.1518
.19	.9992886	.0024615	.0225869	.0563478	.1120	.1318534	.1393
3.20	.9993129	.0023841	.0220290	.0552346	.1107	.1331855	−.1271
.21	.9993363	.0023089	.0214821	.0541346	.1093	.1343969	.1152
.22	.9993590	.0022358	.0209462	.0530481	.1080	.1354907	.1036
.23	.9993810	.0021649	.0204211	.0519751	.1066	.1364699	.0923
.24	.9994024	.0020960	.0199067	.0509158	.1052	.1373374	.0813
.25	.9994230	.0020290	.0194028	.0498702	.1039	.1380961	.0705
.26	.9994429	.0019641	.0189092	.0488384	.1025	.1387491	.0601
.27	.9994623	.0019010	.0184260	.0478205	.1011	.1392992	.0500
.28	.9994810	.0018397	.0179528	.0468165	.0997	.1397494	.0401
.29	.9994991	.0017803	.0174896	.0458265	.0983	.1401026	.0306
3.30	.9995166	.0017226	.0170362	.0448505	.0969	.1403617	−.0213
.31	.9995335	.0016666	.0165925	.0438886	.0955	.1405295	.0123
.32	.9995499	.0016122	.0161584	.0429407	.0941	1406089	−.0036
.33	.9995658	.0015595	.0157337	.0420068	.0927	.1406030	+.0048
.34	.9995811	.0015084	.0153182	.0410870	.0913	.1405139	.0129
.35	.9995959	.0014587	.0149119	.0401813	.0899	.1403450	.0208
.36	.9996103	.0014106	.0145145	.0392896	.0885	.1400989	.0284
.37	.9996242	.0013639	.0141260	.0384160	.0871	.1397782	.0357
.38	.9996376	.0013187	.0137462	.0375482	.0857	.1393857	.0427
.39	.9996505	.0012748	.0133750	.0366984	.0843	.1389240	.0495
3.40	.9996631	.0012322	.0130122	. 358625	.0829	.1383958	+.0561
.41	.9996752	.0011910	.0126577	.0350405	.0815	.1378035	.0623
.42	.9996869	.0011510	.0123164	.0342462	.0801	.1371499	.0684
.43	.9996982	.0011122	.0119730	.0334376	.0788	.1364373	.0741
.44	.9997091	.0010747	.0116426	.0326567	.0774	.1356683	.0796
.45	.9997197	.0010383	.0113199	.0318894	.0761	.1348453	.0849
.46	.9997299	.0010030	.0110047	.0311355	.0747	.1339706	.0900
.47	.9997398	.0009689	.0106971	.0303951	.0734	.1330468	.0948
.48	.9997493	.0009358	.0103968	.0296679	.0721	.1320759	.0994
.49	.9997585	.0009037	.0101037	.0289539	.0707	.1310604	.1037
3.50	.9997674	.0008727	.0098177	.0282531	.0694	.1300024	+.1078
$-X$	$1-\phi_{-1}$	ϕ_0	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6

X	ϕ_{-1}	ϕ_0	ϕ_2	$-\phi_3$	ϕ_4	$-\phi_5$	ϕ_6
3.50	.9997674	.0008727	.0098177	.0282531	.0694	.1300024	.1078
.51	.9997759	.0008426	.0095386	.0275653	.0681	.1289042	.1118
.52	.9997842	.0008135	.0092663	.0268903	.0669	.1277679	.1155
.53	.9997922	.0007853	.0090008	.0262281	.0656	.1265956	.1190
.54	.9997999	.0007581	.0087417	.0255786	.0643	.1253892	.1223
.55	.9998074	.0007317	.0084891	.0249416	.0631	.1241510	.1254
.56	.9998146	.0007061	.0082429	.0243171	.0618	.1228827	.1283
.57	.9998215	.0006814	.0080028	.0237048	.0606	.1215864	.1310
.58	.9998282	.0006575	.0077687	.0231047	.0594	.1202638	.1335
.59	.9998347	.0006343	.0075406	.0225166	.0582	.1189169	.1358
3.60	.9998409	.0006119	.0073183	.0219404	.0570	.1175474	.1380
.61	.9998469	.0005902	.0071018	.0213759	.0559	.1161571	.1400
.62	.9998527	.0005693	.0068908	.0208231	.0547	.1147476	.1419
.63	.9998583	.0005490	.0066853	.0202817	.0536	.1132206	.1435
.64	.9998637	.0005294	.0064851	.0197517	.0524	.1118778	.1450
.65	.9998689	.0005105	.0062902	.0192329	.0513	.1104205	.1464
.66	.9998739	.0004921	.0061004	.0187251	.0502	.1089505	.1476
.67	.9998787	.0004744	.0059157	.0182282	.0492	.1074691	.1487
.68	.9998834	.0004573	.0057358	.0177420	.0481	.1059778	.1496
.69	.9998879	.0004408	.0055608	.0172664	.0470	.1044779	.1504
3.70	.9998922	.0004248	.0053905	.0168013	.0460	.1029708	.1510
.71	.9998964	.0004093	.0052247	.0163465	.0450	.1014578	.1516
.72	.9999004	.0003944	.0050635	.0159019	.0440	.0999402	.1520
.73	.9999043	.0003800	.0049067	.0154672	.0430	.0984191	.1522
.74	.9999080	.0003661	.0047541	.0150424	.0420	.0968958	.1524
.75	.9999116	.0003526	.0046058	.0146272	.0410	.0953713	.1525
.76	.9999150	.0003396	.0044615	.0142216	.0401	.0938467	.1524
.77	.9999184	.0003271	.0043213	.0138254	.0392	.0923232	.1523
.78	.9999216	.0003149	.0041850	.0134384	.0382	.0908017	.1520
.79	.9999247	.0003032	.0040525	.0130605	.0373	.0892831	.1517
3.80	.9999277	.0002919	.0039238	.0126915	.0365	.0877685	.1512
.81	.9999305	.0002810	.0037987	.0123313	.0356	.0862587	.1507
.82	.9999333	.0002705	.0036771	.0119797	.0347	.0847545	.1501
.83	.9999359	.0002604	.0035590	.0116366	.0339	.0832568	.1494
.84	.9999385	.0002506	.0034444	.0113019	.0331	.0817664	.1487
.85	.9999409	.0002411	.0033330	.0109753	.0323	.0802840	.1478
.86	.9999433	.0002320	.0032248	.0106567	.0315	.0788104	.1469
.87	.9999456	.0002232	.0031198	.0103460	.0307	.0773461	.1459
.88	.9999478	.0002147	.0030179	.0100431	.0299	.0758919	.1449
.89	.9999499	.0002065	.0029189	.0097477	.0292	.0744484	.1438
3.90	.9999519	.0001987	.0028229	.0094598	.0284	.0730162	.1426
.91	.9999539	.0001910	.0027297	.0091792	.0277	.0715958	.1414
.92	.9999557	.0001837	.0026393	.0089057	.0270	.0701877	.1402
.93	.9999575	.0001766	.0025516	.0086393	.0263	.0687925	.1389
.94	.9999593	.0001698	.0024665	.0083797	.0256	.0674105	.1375
.95	.9999609	.0001633	.0023840	.0081269	.0249	.0660423	.1361
.96	.9999625	.0001569	.0023039	.0078807	.0243	.0646883	.1347
.97	.9999641	.0001508	.0022263	.0076409	.0237	.0633487	.1332
.98	.9999655	.0001449	.0021511	.0074075	.0230	.0620240	.1317
.99	.9999670	.0001393	.0020781	.0071803	.0224	.0607146	.1302
4.00	.9999683	.0001338	.0020075	.0069592	.0218	.0594206	.1286
$-X$	$1-\phi_{-1}$	ϕ_0	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6

Table 43.

	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	·9048	·8187	·7408	·6703	·6065	·5488	·4966	·4493	·4066	·3679
1	·0905	·1637	·2222	·2681	·3033	·3293	·3476	·3595	·3659	·3679
2	·0045	·0164	·0333	·0536	·0758	·0988	·1217	·1438	·1647	·1839
3	·0002	·0011	·0033	·0072	·0126	·0198	·0284	·0383	·0494	·0613
4		·0001	·0003	·0007	·0016	·0030	·0050	·0077	·0111	·0153
5				·0001	·0002	·0004	·0007	·0012	·0020	·0031
6							·0001	·0002	·0003	·0005
7										·0001

	1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	2,0
0	·3329	·3012	·2725	·2466	·2231	·2019	·1827	·1653	·1496	·1353
1	·3662	·3614	·3543	·3452	·3347	·3230	·3106	·2975	·2842	·2707
2	·2014	·2169	·2303	·2417	·2510	·2584	·2640	·2678	·2700	·2707
3	·0738	·0867	·0998	·1128	·1255	·1378	·1496	·1607	·1710	·1804
4	·0203	·0260	·0324	·0395	·0471	·0551	·0636	·0723	·0812	·0902
5	·0045	·0062	·0084	·0111	·0141	·0176	·0216	·0260	·0309	·0361
6	·0008	·0012	·0018	·0026	·0035	·0047	·0061	·0078	·0098	·0120
7	·0001	·0002	·0003	·0005	·0008	·0011	·0015	·0020	·0027	·0034
8			·0001	·0001	·0001	·0002	·0003	·0005	·0006	·0009
9							·0001	·0001	·0001	·0002

	2,1	2,2	2,3	2,4	2,5	2,6	2,7	2,8	2,9	3,0
0	·1225	·1108	·1003	·0907	·0821	·0743	·0672	·0608	·0550	·0498
1	·2572	·2438	·2306	·2177	·2052	·1931	·1815	·1703	·1596	·1494
2	·2700	·2681	·2652	·2613	·2565	·2510	·2450	·2384	·2314	·2240
3	·1890	·1966	·2033	·2090	·2138	·2176	·2205	·2225	·2237	·2240
4	·0992	·1082	·1169	·1254	·1336	·1414	·1488	·1557	·1622	·1680
5	·0417	·0476	·0538	·0602	·0668	·0735	·0804	·0872	·0940	·1008
6	·0146	·0174	·0206	·0241	·0278	·0319	·0362	·0407	·0455	·0504
7	·0044	·0055	·0068	·0083	·0099	·0118	·0139	·0163	·0188	·0216
8	·0011	·0015	·0019	·0025	·0031	·0038	·0047	·0057	·0068	·0081
9	·0003	·0004	·0005	·0007	·0009	·0011	·0014	·0018	·0022	·0027
10	·0001	·0001	·0001	·0002	·0002	·0003	·0004	·0005	·0006	·0008
11						·0001	·0001	·0001	·0002	·0002
12										·0001

	3,1	3,2	3,3	3,4	3,5	3,6	3,7	3,8	3,9	4,0
0	·0450	·0408	·0369	·0334	·0302	·0273	·0247	·0224	·0202	·0183
1	·1397	·1304	·1217	·1135	·1057	·0984	·0915	·0850	·0789	·0733
2	·2165	·2087	2008	·1929	·1850	·1771	·1692	·1615	·1539	·1465
3	·2237	·2226	·2209	·2186	·2158	·2125	·2087	·2046	·2001	·1954
4	·1733	·1781	·1823	·1858	·1888	·1912	·1931	·1944	·1951	·1954
5	·1075	·1140	·1203	·1264	·1322	·1377	·1429	·1477	·1522	·1563
6	·0555	·0608	·0662	·0716	·0771	·0826	·0881	·0936	·0989	·1042
7	·0246	·0278	·0312	·0348	·0385	·0425	·0466	·0508	·0551	·0595
8	·0095	·0111	·0129	·0148	·0169	·0191	·0215	·0241	·0269	·0298
9	·0033	·0040	·0047	·0056	·0066	·0076	·0089	·0102	·0116	·0132
10	·0010	·0013	·0016	·0019	·0023	·0028	·0033	·0039	·0045	·0053
11	·0003	·0004	·0005	·0006	·0007	·0009	·0011	·0013	·0016	·0019
12	·0001	·0001	·0001	·0002	·0002	·0003	·0003	·0004	·0005	·0006
13					·0001	·0001	·0001	·0001	·0002	·0002
14										·0001

	4,1	4,2	4,3	4,4	4,5	4,6	4,7	4,8	4,9	5,0
0	·0166	·0150	·0136	·0123	·0111	·0101	·0091	·0082	·0074	·0067
1	·0679	·0630	·0583	·0540	·0500	·0462	·0427	·0395	·0365	·0337
2	·1393	·1323	·1254	·1188	·1125	·1063	·1005	·0948	·0894	·0842
3	·1904	·1852	·1798	·1743	·1687	·1631	·1574	·1517	·1460	·1404
4	·1951	·1944	·1933	·1917	·1898	·1875	·1849	·1820	·1789	·1755
5	·1600	·1633	·1662	·1687	·1708	·1725	·1738	·1747	·1753	·1755
6	·1093	·1143	·1191	·1237	·1281	·1323	·1362	·1398	·1432	·1462
7	·0640	·0686	·0732	·0778	·0824	·0869	·0914	·0959	·1002	·1044
8	·0328	·0360	·0393	·0428	·0463	·0500	·0537	·0575	·0614	·0653
9	·0150	·0168	·0188	·0209	·0232	·0255	·0281	·0307	·0334	·0363
10	·0061	·0071	·0081	·0092	·0104	·0118	·0132	·0147	·0164	·0181
11	·0023	·0027	·0032	·0037	·0043	·0049	·0056	·0064	·0073	·0082
12	·0008	·0009	·0011	·0013	·0016	·0019	·0022	·0026	·0030	·0034
13	·0002	·0003	·0004	·0005	·0006	·0007	·0008	·0009	·0011	·0013
14	·0001	·0001	·0001	·0001	·0002	·0002	·0003	·0003	·0004	·0005
15					·0001	·0001	·0001	·0001	0001	·0002

	5,1	5,2	5,3	5,4	5,5	5,6	5,7	5,8	5,9	6,0
0	·0061	·0055	·0050	·0045	·0041	·0037	·0033	·0030	·0027	·0025
1	·0311	·0287	·0265	·0244	·0225	·0207	·0191	·0176	·0162	·0149
2	·0793	·0746	·0701	·0659	·0618	·0580	·0544	·0509	·0477	·0446
3	·1348	·1293	·1239	·1185	·1133	·1082	·1033	·0985	·0938	·0892

	5,1	5,2	5,3	5,4	5,5	5,6	5,7	5,8	5,9	6,0
4	·1719	·1681	·1641	·1600	·1558	·1515	·1472	·1428	·1383	·1339
5	·1753	·1748	·1740	·1728	·1714	·1697	·1678	·1656	·1632	·1606
6	·1490	·1515	·1537	·1555	·1571	·1584	·1594	·1601	·1605	·1606
7	·1086	·1125	·1163	·1200	·1234	·1267	·1298	·1326	·1353	·1377
8	·0692	·0731	·0771	·0810	·0849	·0887	·0925	·0962	·0998	·1033
9	·0392	·0423	·0454	·0486	·0519	·0552	·0586	·0620	·0654	·0688
10	·0200	·0220	·0241	·0262	·0285	·0309	·0334	·0359	·0386	·0413
11	·0093	·0104	·0116	·0129	·0143	·0157	·0173	·0190	·0207	·0225
12	·0039	·0045	·0051	·0058	·0065	·0073	·0082	·0092	·0102	·0115
13	·0015	·0018	·0021	·0024	·0028	·0032	·0036	·0041	·0046	·0052
14	·0006	·0007	·0008	·0009	·0011	·0013	·0015	·0017	·0019	·0022
15	·0002	·0002	·0003	·0003	·0004	·0005	·0006	·0007	·0008	·0009
16	·0001	·0001	·0001	·0001	·0001	·0002	·0002	·0002	·0003	·0003
17						·0001	·0001	·0001	·0001	·0001

	6,1	6,2	6,3	6,4	6,5	6,6	6,7	6,8	6,9	7,0
0	·0022	·0020	·0018	·0017	·0015	·0014	·0012	·0011	·0010	·0009
1	·0137	·0126	·0116	·0106	·0098	·0090	·0082	·0076	·0070	·0064
2	·0417	·0390	·0364	·0340	·0318	·0296	·0276	·0258	·0240	·0223
3	·0848	·0806	·0765	·0726	·0688	·0652	·0617	·0584	·0552	·0521
4	·1294	·1249	·1205	·1162	·1118	·1076	·1034	·0992	·0952	·0912
5	·1579	·1549	·1519	·1487	·1454	·1420	·1385	·1349	·1314	·1277
6	·1605	·1601	·1595	·1586	·1575	·1562	·1546	·1529	·1511	·1490
7	·1399	·1418	·1435	·1450	·1462	·1472	·1480	·1486	·1489	·1490
8	·1066	·1099	·1130	·1160	·1188	·1215	·1240	·1263	·1284	·1304
9	·0723	·0757	·0791	·0825	·0858	·0891	·0923	·0954	·0985	·1014
10	·0441	·0469	·0498	·0528	·0558	·0588	·0618	·0649	·0679	·0710
11	·0244	·0265	·0285	·0307	·0330	·0353	·0377	·0401	·0426	·0452
12	·0124	·0137	·0150	·0164	·0179	·0194	·0210	·0227	·0245	·0263
13	·0058	·0065	·0073	·0081	·0089	·0099	·0108	·0119	·0130	·0142
14	·0025	·0029	·0033	·0037	·0041	·0046	·0052	·0058	·0064	·0071
15	·0010	·0012	·0014	·0016	·0018	·0020	·0023	·0026	·0029	·0033
16	·0004	·0005	·0005	·0006	·0007	·0008	·0010	·0011	·0013	·0014
17	·0001	·0002	·0002	·0002	·0003	·0003	·0004	·0004	·0005	·0006
18		·0001	·0001	·0001	·0001	·0001	·0001	·0002	·0002	·0002
19							·0001	·0001	·0001	·0001

	7,1	7,2	7,3	7,4	7,5	7,6	7,7	7,8	7,9	8,0
0	·0008	·0007	·0007	·0006	·0006	·0005	·0005	·0004	·0004	·0003
1	·0059	·0054	·0049	·0045	·0041	·0038	·0035	·0032	·0029	·0027
2	·0208	·0194	·0180	·0167	·0156	·0145	·0134	·0125	·0116	·0107
3	·0492	·0464	·0438	·0413	·0389	·0366	·0345	·0324	·0305	·0286
4	·0874	·0836	·0799	·0764	·0729	·0696	·0663	·0632	·0602	0573
5	·1241	·1204	·1167	·1130	.1094	·1057	·1021	·0986	·0951	·0916
6	·1468	·1445	·1420	·1394	·1367	·1339	·1311	·1282	·1252	·1221
7	·1489	·1486	·1481	·1474	·1465	·1454	·1442	·1428	·1413	·1396
8	·1321	·1337	·1351	·1363	·1373	·1381	·1388	·1392	·1395	1396
9	·1042	·1070	·1096	·1121	·1144	·1167	·1187	·1207	·1224	·1241
10	·0740	·0770	·0800	·0829	·0858	·0887	·0914	·0941	·0967	·0993
11	·0478	·0504	·0531	·0558	·0585	·0613	·0640	·0667	·0695	·0722
12	·0283	·0303	·0323	·0344	·0366	·0388	·0411	·0434	·0457	·0481
13	·0154	·0168	·0181	·0196	·0211	·0227	·0243	0260	·0278	·0296
14	·0078	·0086	·0095	·0104	·0113	·0123	·0134	·0145	·0157	·0169
15	·0037	·0041	·0046	·0051	·0057	·0062	·0069	·0075	·0083	·0090
16	·0016	·0019	·0021	·0024	·0026	·0030	·0033	·0037	·0041	·0045
17	·0007	·0008	·0009	·0010	·0012	·0013	·0015	·0017	·0019	·0021
18	·0003	·0003	·0004	·0004	·0005	·0006	·0006	·0007	·0008	·0009
19	·0001	·0001	·0001	·0002	·0002	·0002	·0003	·0003	·0003	·0004
20			·0001	·0001	·0001	·0001	·0001	·0001	·0001	·0002
21									·0001	·0001

	8,1	8,2	8,3	8,4	8,5	8,6	8,7	8,8	8,9	9,0
0	·0003	·0003	·0002	·0002	·0002	·0002	·0002	·0002	·0001	·0001
1	·0025	·0023	·0021	·0019	·0017	·0016	·0014	·0013	·0012	·0011
2	·0100	·0092	·0086	·0079	·0074	·0068	·0063	·0058	·0054	·0050
3	·0269	·0252	·0237	·0222	·0208	·0195	·0183	·0171	·0160	·0150
4	·0544	·0517	·0491	·0466	·0443	·0420	·0398	·0377	·0357	·0337
5	·0882	·0849	·0816	·0784	·0752	·0722	·0692	·0663	0635	·0607
6	·1191	·1160	·1128	·1097	·1066	·1034	·1003	·0972	·0941	·0911
7	·1378	·1358	·1338	·1317	·1294	·1271	·1247	·1222	·1197	·1171
8	·1395	·1392	·1388	·1382	·1375	·1366	·1356	·1344	·1332	·1318
9	·1256	·1269	1280	·1290	·1299	·1306	·1311	·1315	·1317	·1318
10	·1017	·1040	·1063	·1084	·1104	·1123	·1140	·1157	·1172	·1186
11	·0749	·0776	·0802	·0828	·0853	·0878	·0902	·0925	·0948	·0970
12	·0505	·0530	·0555	·0579	·0604	·0629	·0654	·0679	·0703	·0728

	8,1	8,2	8,3	8,4	8,5	8,6	8,7	8,8	8,9	9,0
13	·0315	·0334	·0354	·0374	·0395	·0416	·0438	·0459	·0481	·0504
14	·0182	·0196	·0210	·0225	·0240	·0256	·0272	·0289	·0306	·0324
15	·0098	·0107	·0116	·0126	·0136	·0147	·0158	·0169	·0182	·0194
16	·0050	·0055	·0060	·0066	·0072	·0079	·0086	·0093	·0101	·0109
17	·0024	·0026	·0029	·0033	·0036	·0040	·0044	·0048	·0053	·0058
18	·0011	·0012	·0014	·0015	·0017	·0019	·0021	·0024	·0026	·0029
19	·0005	·0005	·0006	·0007	·0008	·0009	·0010	·0011	·0012	·0014
20	·0002	·0002	·0002	·0003	·0003	·0004	·0004	·0005	·0005	·0006
21	·0001	·0001	·0001	·0001	·0001	·0002	·0002	·0002	·0002	·0003
22					·0001	·0001	·0001	·0001	·0001	·0001

	9,1	9,2	9,3	9,4	9,5	9,6	9,7	9,8	9 9	10,0
0	·0001	·0001	·0001	·0001	·0001	·0001	·0001	·0001	·0001	
1	·0010	·0009	·0009	·0008	·0007	·0007	·0006	·0005	·0005	·0005
2	·0046	·0043	·0040	·0037	·0034	·0031	·0029	·0027	·0025	·0023
3	·0140	·0131	·0123	·0115	·0107	·0100	·0093	·0087	·0081	·0076
4	·0319	·0302	·0285	·0269	·0254	·0240	·0226	·0213	·0201	·0189
5	·0581	·0555	·0530	·0506	·0483	·0460	·0439	·0418	·0398	·0378
6	·0881	·0851	·0822	·0793	·0764	·0736	·0709	·0682	·0656	·0631
7	·1145	·1118	·1091	·1064	·1037	·1010	·0982	·0955	·0928	·0901
8	·1302	·1286	·1269	·1251	·1232	·1212	·1191	·1170	·1148	·1126
9	·1317	·1315	·1311	·1306	·1300	·1293	·1284	·1274	·1263	·1251
10	·1198	·1210	·1219	·1228	·1235	·1241	·1245	·1249	·1250	·1251
11	·0991	·1012	·1031	·1049	·1067	·1083	·1098	·1112	·1125	·1137
12	·0752	·0776	·0799	·0822	·0844	·0866	·0888	·0908	·0928	·0948
13	·0526	·0549	·0572	·0594	·0617	·0640	·0662	·0685	·0707	·0729
14	·0342	·0361	·0380	·0399	·0419	·0439	·0459	·0479	·0500	·0521
15	·0208	·0221	·0235	·0250	·0265	·0281	·0297	·0313	·0330	·0347
16	·0118	·0127	·0137	·0147	·0157	·0168	·0180	·0192	·0204	·0217
17	·0063	·0069	·0075	·0081	·0088	·0095	·0103	·0111	·0119	·0128
18	·0032	·0035	·0039	·0042	·0046	·0051	·0055	·0060	·0065	·0071
19	·0015	·0017	·0019	·0021	·0023	·0026	·0028	·0031	·0034	·0037
20	·0007	·0008	·0009	·0010	·0011	·0012	·0014	·0015	·0017	·0019
21	·0003	·0003	·0004	·0004	·0005	·0006	·0006	·0007	·0008	·0009
22	·0001	·0001	·0002	·0002	·0002	·0002	·0003	·0003	·0004	·0004
23		·0001	·0001	·0001	·0001	·0001	·0001	·0001	·0002	·0002
24								·0001	·0001	·0001

Table 44.

$E =$	$\lvert s \rvert <$
.00	.000
.03	.180
.06	.251
.09	.302
.12	.353
.15	.406
.18	.445
.21	.478
.24	.496
.27	.518
.30	.524
.33	.522
.36	.509
.39	.483
.42	.441
.45	.374
.48	.253
.50	.000

List of Works cited.

1. BORTKIEWICZ, L. V. <*Bortkewitsch* in German literature>, (1898) *Das Gesetz der kleinen Zahlen*, Leipzig

2. CHARLIER, C. V. L., (1905) 2 *Arkiv för Matematik, Astronomi och Fysik* No. 8

3. ---, (1905) 2 *Arkiv för Matematik, Astronomi och Fysik* No. 15

4. ---, (1906) 1 *Acta Universitatis Lundensis*, Abt. 2, Nr. 5

5. ---, (1909) 5 *Arkiv för Matematik, Astronomi och Fysik* No. 15

6. ---, (1912) 8 *Acta Universitatis Lundensis*, Abt. 2, Nr. 2

7. EDGEWORTH, F. Y., (1905) 20 *Proceedings Cambridge Philosophical Society* 36

8. FIBIGER, J., (1898) 6 *Hospitalstidende* 309-325

9. FÜRST, C. M., and F. C. C. HANSEN, (1915) *Crania Groenlandica: a description of Greenland Eskimo Crania, with an Introduction on the Geography and History of Greenland*; Copenhagen, Andr. Fred. Höst & Sön

10. GALTON, F., (1889) *Natural Inheritance*, London

11. HAGEN, G., (1837) *Grundzuge der Wahrscheinlichkeits-Rechnung*, Berlin

12. JØRGENSEN, N. R., (1916) *Undersøgelser over Frequensflader og Korrelation*, København, Arnold Busck

13. JOHANNSEN, W., (1909) *Die Elemente der exakten Erblichkeitslehre*, Jena, Gustav Fischer

14. LAPLACE, P. S., (1812) *Théorie analytique des probabilités*, Paris

15. LEXIS, W. H. R. A., (1877) *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg in Baden

16. LOWAN, A. N., director, (1942) *Tables of Probability Functions*, II, New York, National Bureau of Standards

17. MORANT, G. M., and B. L. WELCH, (1939) *Bibliography of the statistical and other writings of Karl Pearson*, London, University College

18. PEARSON, K., Mathematical contributions to the theory of evolution. Nineteen papers, numbered I-XIX , and eight papers not numbered. For bibliography and criticism see MORANT <17> at *nos*, 2 , 4 , 6 , 16 , 17 , 20 , 27 , 28 , 33 , 34 , 43 , 75 , 76 , 92 , 99 , 111 , 187 , 188 , 240 , for the numbered, and 7 , 8 , 11 , 12 , 18 , 19 , 29 , 61 , for the unnumbered, papers.

19. ---, *ed.*, (1930) *Tables for statisticians and biometricians*, I, 3. Ed., London, Biometric Laboratory

20. ---, *ed.*, (1931) *idem*, II, 1. Ed., *ibid.*

21. RETZIUS, G., and C. M. FÜRST, (1902) *Anthropologia suecica: Beiträge zur Anthropologie der Schweden*, Stockholm

22. WICKSELL, S. D., (1916) 3 *Svenska Aktuarieföreningens Tidskrift* 165-213

23. YULE, G. U., (1900) A 194 *Philosophical Transactions* 257-319

24. ---, (1912) 75 *Journal Royal Statistical Society* 579-642

Index. The numbers refer to articles.